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Quadratic irrationals and their N-continued fraction expansions

Abstract: A real number has a periodic regular continued fraction expansion if and only if it is a quadratic irrational. This classical result is also true for many other continued fraction families. Is it also true for N-continued fractions expansions? These are expansions of the form:

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{\ddots}}} \tag{1}$$

where $N \in \mathbb{N}_{>1}$ and $d_i \in \mathbb{N}$. N-continued fractions and regular continued fractions differ in many ways. For example, N-continued fractions are not unique. In this talk we focus on the following question. For a quadratic irrational x, what can we say about the periodicity of its N-continued fraction expansions?

To find continued fractions of the form (1) we can use dynamical systems. Here we will focus on the following family, for which we show that for certain choices of α there exist quadratic irrationals with an a-periodic expansion. Let $\alpha \in (0, \sqrt{N}-1]$, and define the map $T_{N,\alpha} : [\alpha, \alpha+1] \to [\alpha, \alpha+1)$ as

$$T_{N,\alpha}(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} - \alpha \right\rfloor .$$

The map $T_{N,\alpha}$ generates N-continued fractions with only finitely many different digits. Moreover, $T_{N,\alpha}$ maps quadratic irrationals onto quadratic irrationals. Let $x_0 \in [\alpha, \alpha+1]$ be a quadratic irrational that satisfies $A_0x^2 + B_0x + C_0 = 0$ and define $x_n = T^n_{N,\alpha}(x_0)$ which satisfies $A_nx^2 + B_nx + C_n = 0$ for some $A_n, B_n, C_n \in \mathbb{Z}$. For the determinant of these equations we can show that

$$B_n^2 - 4A_nC_n = N^{2n}(B_0^2 - 4A_0C_0) \tag{2}$$

holds. Now the idea is to pick α and initial coefficients A_0, B_0 and C_0 such that A_n, B_n and C_n are co-prime for all n. In this way (2) gives us the a-periodicity. This talk is joint work with Cor Kraaikamp.