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On numeration systems with positive and negative digits

Abstract: Work in progress and in common with Rita GIULIANO (Pisa) and Labib HADDAD (Paris).

The numeration systems considered in this talk have as *basis* either the powers $\{k^0, k^1, k^2, k^3, \dots\}$ of an integer $k \geq 2$ or, more generally, the so called *Cantor* (or *sterling* !) basis $\{b_0 = 1, b_1 = k_1, \dots, b_{i+1} = b_i k_{i+1}, \dots\}$ where $(k_i)_{i \geq 1}$ is a given sequence of integers with $k_i \geq 2$ for all $i \geq 1$. The *digits* c_i will be integers subject to certain conditions. The system will be *complete* and *non redundant*: every integer $n \in \mathbf{Z}$ will have a unique representation $n = \sum_{i=0}^{\infty} c_i b_i$ where all, except finitely many, c_i are zero, each $c_i \in \mathbf{Z}$ depends on k_{i+1} and is uniquely determined as for its sign and for its absolute value $|c_i|$. We prove:

Theorem 1.- Given a Cantor basis $(b_i)_{i \geq 0}$, every nonzero integer n has a unique representation of the form $n = -c_1 b_{m_1} + c_2 b_{m_2} - \cdots + (-1)^j c_j b_{m_j} + \cdots + (-1)^s c_s b_{m_s}$ where $1 \leq c_j < k_{m_j+1}$ for every index j and the s integers m_j form a strictly increasing sequence $0 \leq m_1 < m_2 < \cdots < m_s$.

Notation: Let $a > 0$ be an integer. We define a set of *residues*

$R(a) = \{-a/2, -a/2 + 1, \dots, -1, 0, 1, \dots, a/2\}$ if a is even;

$R(a) = \{-(a-1)/2, \dots, -1, 0, 1, \dots, (a+1)/2\}$ if a is odd.

Theorem 2.- Given a Cantor basis $(b_i)_{i \geq 0}$, every nonzero integer n has a unique representation of the form $n = c_0 b_0 + c_1 b_1 + \cdots + c_s b_s$ where, for every i, $0 \leq i \leq s$, $c_i \in R(k_{i+1})$, $c_s \neq 0$, and the c_i's are subject to the following condition: If one of the c_i's with $i < s$ is an extreme point of the interval $R(k_{i+1})$, then $c_i c_{i+1} \geq 0$ and c_{i+1} is not an extreme point of the interval $R(k_{i+2})$.

The end of the talk will be devoted to some open questions concerning the cost function $C(n) = \sum_{i=0}^{\infty} |c_i|$.