MEAN VALUES OF SOME ARITHMETICAL FUNCTIONS OVER FRIABLE INTEGERS

Abstract. Let \( S(x, y) \) be the set of integers up to \( x \), all of whose prime factors are \( \leq y \), and \( s_q(n) \) be the sum-of-digits function in base \( (q \geq 2) \) of the positive integer \( n \). Our main result is to estimate the sum \( \sum_{n \in S(x,x)} \omega(n) \), where \( \omega(n) \) is either \( \Omega(n) \) or \( \Omega_1(n) \), the number of distinct prime factors and the total number of prime factors \( p \) of a positive integer \( n \), such that \( s_q(p) \equiv a \mod b \), \( a, b \in \mathbb{Z} \).

Plan for today:

1. Fiable integers.
2. Mean values of \( \omega(n) \) and \( \omega^*(n) \).
3. Fiable mean value of \( \omega(n) \).
4. Our results.
5. Perspectives.

**Definition [\( \gamma \)-fiable integer]**

An integer \( n \) \( \geq 1 \) is called \( \gamma \)-fiable if \( P(n) \leq \gamma \).
- We denote by \( S(\gamma) \) the set of integers that are \( \gamma \)-fiable.
- We denote by \( S(x, \gamma) \) the set of integers less than or equal to \( x \) that are \( \gamma \)-fiable.
- We denote by \( \Phi(x, \gamma) \) the counting function elements in \( S(x, \gamma) \) :

\[ \Phi(x, \gamma) = \#S(x, \gamma) \]

Hildebrand and Tenenbaum estimate \( \Psi(x, \gamma) \) for a wide range of parameters of \( x \) and \( \gamma \), using a special point they call a "saddle point" \( x_{\gamma}(x) \). They have established the following asymptotic :

\[ \Psi(x, \gamma) = e^{\gamma(x)}(1 + O(1))(\log \gamma + O(1)) \log x \]

Where \( \gamma = \log \log x \), uniformly for \( 2 \leq \gamma \leq x \), and \( a = x_{\gamma}(x) = 1 - \frac{1}{\log(\log(\log(x)) + 1)} + O(1) \log x \),

\[ \text{when } \log x < \gamma \leq x \]

Obtaining asymptotic results that provide evaluations for sums of arithmetic functions \( f \) of the form \( \sum_{n \leq x, \gamma(n)} f(n) \), is pivotal in number theory. The growing importance of fiable integers naturally drew certain authors to obtain results regarding the mean values of arithmetical functions over this set of integers. In other words, the asymptotic behavior of the summatory function \( \sum_{n \leq x, \gamma(n)} f(n) \), is the "fiable mean-value" of the function \( f \). In this presentation we delve into the additif functions:

\[ \omega(n) = \sum_{d|\gamma(n)} 1, \text{ and } \Omega(n) = \sum_{d|\gamma(n)} \log \gamma \]

\[ \omega^*(n) = \sum_{d|\gamma(n)} \log \gamma, \text{ and } \Omega^*(n) = \sum_{d|\gamma(n)} \log \gamma \]

\( s_q(n) \) is the sum of digit's function of the integer \( n \) in the base \( q \).

Mean values of \( \omega(n) \) and \( \omega^*(n) \)

Hardy showed that:

\[ \sum_{n \leq x} \omega(n) = \omega(x) + O(x^{1/2}) \]

\[ \sum_{n \leq x} \omega^*(n) = \omega^*(x) + O(x^{1/2}) \]

Our results

We point out a few remarks:

1. Theorem 1: The average number of prime divisors of \( n \) is \( \log \log x \). Further development was made by Erdos and Kar, they made a stronger result than of the Hardy-Ramanujan Theorem in a completely different way, they proved a distribution result via the use of probabilistic tools, namely, the existence of a normal distribution for \( \omega(n) \).
2. More precisely, they proved that for \( x, \gamma \in \mathbb{R} \):

\[ \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \omega(n) = \frac{1}{2} \log \log x, \text{ and } \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \omega^*(n) = \frac{1}{2} \log \log x \]

Perspectives

- Hardy and Ramanujan showed that the normal order of \( \omega(n) \) is \( \log \log x \). Further development was made by Erdos and Kar, they made a stronger result than of the Hardy-Ramanujan Theorem in a completely different way, they proved a distributional result via the use of probabilistic tools, namely, the existence of a normal distribution for \( \omega(n) \).

**Our main result**

For any \( x > 1 \) and any real \( \gamma \), we have:

\[ \sum_{n \leq x} \omega(n) = \omega(x) + O(x^{1/2}) \]

\[ \sum_{n \leq x} \omega^*(n) = \omega^*(x) + O(x^{1/2}) \]

\[ \omega(n) \text{ always equals to a normal order } \Omega(n) \]

\[ \omega^*(n) \text{ always equals to a normal order } \Omega^*(n) \]

Perspectives

- Instead of the sequence of all natural numbers, Hildebrand considered the case of the \( \gamma \)-fiable integers. In this case, he proved that:

\[ \sum_{n \leq x, \gamma(n)} \omega(n) = \frac{1}{2} \log \log x \]

\[ \sum_{n \leq x, \gamma(n)} \omega^*(n) = \frac{1}{2} \log \log x \]

- This holds when \( x = \log \log x \), where, as always, \( \gamma = \log \log x \).

Thank you for your attention!