Mean values of some arithmetical functions over friable integers

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4 juin 2024

Plan for today :

1. Friable integers.
2. Mean value of $\tilde{\omega}(n)$.
3. Firable mean value of $\omega(n)$.
4. Our results.

Definition [$y-$friable integer] :

An integer $n \geq 1$ is called y-friable if $P(n) \leq y$.

• We denote by $S(y)$ the set of integers that are y-friable.

$$S(y) = \{n \in \mathbb{N}^* : n \text{ is } \quad y\text{-friable}\}.$$

• We denote by $S(x, y)$ the set of integers less than or equal to x that are y-friable,

$$S(x, y) = \{1 \leq n \leq x : n \text{ is } \; y\text{-friable}\}.$$

• We denote by $\Psi(x, y)$ the counting function of elements in $S(x, y)$:

$$\Psi(x, y) = \#S(x, y).$$

Hildebrand and Tenenbaum estimate $\Psi(x,y)$ for a wide range of parameters of x and y, using a special point they call a "saddle point" $\alpha(x,y)$. They have established the following asymptotic :

$$\Psi(x,y) = \frac{x^{\alpha}\zeta(\alpha,y)}{\alpha\sqrt{2\pi(1+(\log x)/y)\log x\log y}}\Big(1+O\Big(\frac{1}{\log(u+1)} + \frac{1}{\log y}\Big)\Big).$$

Where $u = \frac{\log x}{\log y}$, uniformly for $2 \leq y \leq x$, and

$$\alpha := \alpha(x,y) = 1 - \frac{\log(u(\log(u+1))}{\log y} + O\left(\frac{1}{\log y}\right).$$

when $\log x < y \leq x$.

Obtaining asymptotic results that provide evaluations for sums of arithmetic functions f of the form $\sum_{n\leq x} f(n)$, is pivotal in number theory. The growing importance of friable integers naturally drove certain authors to obtain results regarding the mean values of arithmetic functions over this set of integers. In other words the asymptotic behavior of the summatory function $\sum_{n\in S(x,y)} f(n)$, we call it the "friable mean-value" of the function f. In this presenation we delve into the additive functions :

$$\omega(n) = \sum_{p|n} 1, \qquad \text{and} \qquad \Omega(n) = \sum_{p^k|n \;\; k\geq 1} 1,$$

$$\tilde{\omega}(n) = \sum_{\substack{p|n \\ s_q(p)\equiv a \mod b}} 1, \qquad \text{and} \qquad \tilde{\Omega}(n) = \sum_{\substack{p^k|n \;\; k\geq 1 \\ s_q(p)\equiv a \mod b}} 1,$$

$s_q(n)$ is the sum of digits function of the integer n in the base q.

Further, Mkaouar and Wannès established the arithmetic mean of the functions $\tilde{\omega}(n)$ and $\tilde{\Omega}(n)$.

$$\sum_{n\leq x}\tilde{\omega}(n)=\frac{x}{b}\log\log x+\beta x+O\Big(\frac{x}{\log x}\Big),$$

where η is Mertens's constant, and

$$\beta=\frac{1}{b}\sum_{j=1}^{b-1}e\Big(\frac{-aj}{b}\Big)\int_2^{+\infty}\Big(\sum_{p\leq t}e\Big(\frac{j}{b}s_q(p)\Big)\Big)\frac{dt}{t^2}+\eta b^{-1}.$$

In this direction, we turn our focus to the case of friable integers, particularly, we estimate the sums :

$$\sum_{n\in S(x,y)}\tilde{\omega}(n),\qquad \text{and}\qquad \sum_{n\in S(x,y)}\tilde{\Omega}(n).$$

We shall first recall the friable mean-value of $\omega(n)$ which is natural to consider at a first pass.

3. Friable mean values of $\omega(n)$

One notable estimation, presented by Mehdizadeh he proved

$$\sum_{n\in S(x,y)} \omega(n) = \Psi(x,y)\{M_y + O(1)\}, \tag{1}$$

where

$$M_y = \log\log y + \frac{uy}{y+\log x}\left\{1 + O\Big(\frac{1}{\log y} + \frac{1}{\log 2u}\Big)\right\},$$

uniformly for $2 \le y \le x$.

4. Our Results

Now we are able to state our results, we firstly look for the friable mean-value of the function $\tilde{\omega}(n)$.

Theorem 1 :

We have the asymptotic

$$\sum_{n\in S(x,y)} \tilde{\omega}(n) = \Psi(x,y)(\frac{1}{b}M_y + \delta)\left\{1 + O\Big(\frac{1}{u}\Big)\right\},$$

where

$$\delta = \frac{1}{b}\sum_{j=1}^{b-1} e\Big(\frac{-aj}{b}\Big)\int_2^{+\infty}\Big(\sum_{p\le t} e\Big(\frac{j}{b}s_q(p)\Big)\Big)\frac{dt}{t^{\alpha+1}}. \qquad (2)$$

This is uniformly in the range

$$x \geq x_0, \qquad \log^{b^2/c_1} x < y \leq x, \tag{3}$$

where the constant c_1 is a constant.

Thus, our result implies that $\omega(n)$ is equidistributed in the congruence classes $s_q(p) \equiv a \mod b$ in the range (3).

• Theorem 1 shows that the average number of prime divisors of n such that $n \in S(x,y)$ and satisfying the congruence relation $s_q(p) \equiv a \mod b$ is M_y/b, valid in the range (3). Next, we aim to show a better result, precisely that almost all integers in $S(x,y)$ have $M_y/b(1+o(1))$ prime factors, always respecting the same conditions. But before we delve into the proofs, we shall first recall the concept of the "*normal order*" of a function.

Definition [Normal order] :

An arithmetic function $g(n)$ is said to have a normal order $G(n)$ if, for any $\varepsilon > 0$, for almost all $n \leq x$, one has

$$(1-\varepsilon)G(n) \leq g(n) \leq (1+\varepsilon)G(n). \quad (4)$$

This means that the proportion of $n \leq x$ for which equation (4) does not hold tends to 0 as x tends to infinity.

• An eminent example of this concept is the Hardy-Ramanujan theorem, asserting that the normal order of the function $\omega(n)$ equals $\log\log n$. Analogous to the Hardy-Ramanujan theorem, we delve into the pursuit of the friable normal order for the function $\tilde{\omega}(n)$. The Turán Theorem concerns the second moment of $\omega(n)$ and implies the Hardy-Ramanujan Theorem. Following the steps of Turán's proof, we seek an asymptotic for $\sum_{n\in S(x,y)} \tilde{\omega}(n)^2$. We present the next theorem

Theorem 2 :

We have the asymptotic

$$\sum_{n\in S(x,y)} \tilde{\omega}(n)^2 = \Psi(x,y)\left\{1+O\Big(\frac{1}{u}\Big)\right\}\left\{\frac{1}{b^2}M_y^2+O(1)\right\},$$

uniformly in the range (3), δ is defined in (2) and c_1 is a constant.

Consequently, we deduce our main result

Theorem 3 :

For any $\varepsilon > 0$, we have

$$\lim_{x\to\infty} \frac{\#\left\{n \in S(x,y) : (1-\varepsilon)\frac{M_y}{b} \leq \tilde{\omega}(n) \leq (1+\varepsilon)\frac{M_y}{b}\right\}}{\Psi(x,y)} = 1,$$

uniformly in the range (3), and the constant c_1 is a constant.

Finally, as an application of Theorem 1, we deduce

Theorem 4 :

We have the asymptotic

$$\sum_{n\in S(x,y)} \tilde{\Omega}(n) = \Psi(x,y) M_y \left\{\frac{1}{b} + O\Big(\frac{1}{u}\Big)\right\} + O(\Psi(x,y)).$$