Morphic sequences: characterization, visualization and equality

Hans Zantema

Technische Universiteit Eindhoven and Radboud Universiteit Nijmegen
(until Sept 2022)

Numeration, June, 2024

Sequences

Sequences

A(n infinite) sequence over an alphabet A is a map $\sigma : \mathbb{N} \to A$

Sequences

A(n infinite) sequence over an alphabet A is a map $\sigma : \mathbb{N} \to A$

$$\sigma \;=\; \sigma(0)\sigma(1)\sigma(2)\sigma(3)\cdots$$

A(n infinite) sequence over an alphabet A is a map $\sigma : \mathbb{N} \to A$

$$\sigma \;=\; \sigma(0)\sigma(1)\sigma(2)\sigma(3)\cdots$$

A sequence is *ultimately periodic* if there exist k, n such that $\sigma(i + k) = \sigma(i)$ for all $i \geq n$

Sequences

A(n infinite) sequence over an alphabet A is a map $\sigma : \mathbb{N} \to A$

$$\sigma \ = \ \sigma(0)\sigma(1)\sigma(2)\sigma(3)\cdots$$

A sequence is *ultimately periodic* if there exist k, n such that $\sigma(i + k) = \sigma(i)$ for all $i \geq n$

These are boring

A(n infinite) sequence over an alphabet A is a map $\sigma : \mathbb{N} \to A$

$$\sigma \ = \ \sigma(0)\sigma(1)\sigma(2)\sigma(3)\cdots$$

A sequence is *ultimately periodic* if there exist k, n such that $\sigma(i+k) = \sigma(i)$ for all $i \geq n$

These are boring

What are the simplest sequences that are *not* ultimately periodic?

Morphisms, morphic sequences

Morphisms, morphic sequences

A *morphism* $f : A \to A^+$ can be extended to strings and to sequences

Morphisms, morphic sequences

A *morphism* $f : A \to A^+$ can be extended to strings and to sequences

If $f(a) = au$, $u \neq \epsilon$, then f has a *unique fixed point* starting in a:

Morphisms, morphic sequences

A *morphism* $f : A \to A^+$ can be extended to strings and to sequences

If $f(a) = au$, $u \neq \epsilon$, then f has a *unique fixed point* starting in a:

$$f(f(a)) = f(au) = f(a)f(u) = auf(u)$$

A *morphism* $f : A \to A^+$ can be extended to strings and to sequences

If $f(a) = au$, $u \neq \epsilon$, then f has a *unique fixed point* starting in a:

$$f(f(a)) = f(au) = f(a)f(u) = auf(u)$$

$$f(f(f(a))) = f(auf(u)) = auf(u)f^2(u)$$

Morphisms, morphic sequences

A *morphism* $f : A \to A^+$ can be extended to strings and to sequences

If $f(a) = au$, $u \neq \epsilon$, then f has a *unique fixed point* starting in a:

$$f(f(a)) = f(au) = f(a)f(u) = auf(u)$$

$$f(f(f(a))) = f(auf(u)) = auf(u)f^2(u)$$

$$\cdots\cdots$$

$$f^\infty(a) = auf(u)f^2(u)f^3(u)f^4(u)\cdots$$

Morphisms, morphic sequences

A *morphism* $f : A \to A^+$ can be extended to strings and to sequences

If $f(a) = au$, $u \neq \epsilon$, then f has a *unique fixed point* starting in a:

$$f(f(a)) = f(au) = f(a)f(u) = auf(u)$$

$$f(f(f(a))) = f(auf(u)) = auf(u)f^2(u)$$

$$\cdots\cdots$$

$$f^\infty(a) = auf(u)f^2(u)f^3(u)f^4(u)\cdots$$

Such a sequence is called *pure morphic*

Morphisms, morphic sequences

A *morphism* $f : A \to A^+$ can be extended to strings and to sequences

If $f(a) = au$, $u \neq \epsilon$, then f has a *unique fixed point* starting in a:

$$f(f(a)) = f(au) = f(a)f(u) = auf(u)$$

$$f(f(f(a))) = f(auf(u)) = auf(u)f^2(u)$$

$$\cdots\cdots$$

$$f^\infty(a) = auf(u)f^2(u)f^3(u)f^4(u)\cdots$$

Such a sequence is called *pure morphic*

If $\tau : B \to A$ (called a *coding*) and σ is pure morphic over B, then $\tau(\sigma)$ is called *morphic*

Example

Example

The *Thue-Morse* sequence

$$\mathbf{t} = 0110100110010110 \cdots$$

is defined by $\mathbf{t} \ = \ f^{\infty}(0)$ for $f(0) = 01$, $f(1) = 10$

Example

The *Thue-Morse* sequence

$$\mathbf{t} = 0110100110010110\cdots$$

is defined by $\mathbf{t} \;=\; f^{\infty}(0)$ for $f(0) = 01$, $f(1) = 10$

The *Fibonacci* sequence

$$\text{fib} = 0100101001001\cdots$$

is defined by $\text{fib} \;=\; f^{\infty}(0)$ for $f(0) = 01$, $f(1) = 0$

Example

The *Thue-Morse* sequence

$$\mathbf{t} = 0110100110010110 \cdots$$

is defined by $\mathbf{t} \ = \ f^{\infty}(0)$ for $f(0) = 01$, $f(1) = 10$

The *Fibonacci* sequence

$$\mathrm{fib} = 0100101001001 \cdots$$

is defined by $\mathrm{fib} \ = \ f^{\infty}(0)$ for $f(0) = 01$, $f(1) = 0$

These are pure morphic

Example

The *Thue-Morse* sequence

$$\mathbf{t} = 0110100110010110\cdots$$

is defined by $\mathbf{t} \ = \ f^{\infty}(0)$ for $f(0) = 01,\ f(1) = 10$

The *Fibonacci* sequence

$$\mathrm{fib} = 0100101001001\cdots$$

is defined by $\mathrm{fib} \ = \ f^{\infty}(0)$ for $f(0) = 01,\ f(1) = 0$

These are pure morphic

For $f(0) = 0,\ f(1) = 10,\ f(2) = 210,\ \tau(0) = 0,\ \tau(1) = \tau(2) = 1$
we obtain

$$\tau(f^{\infty}(2)) \ = \ 11010010^{3}10^{4}10^{5}1\cdots$$

Example

The *Thue-Morse* sequence

$$\mathbf{t} = 0110100110010110 \cdots$$

is defined by $\mathbf{t} \; = \; f^{\infty}(0)$ for $f(0) = 01$, $f(1) = 10$

The *Fibonacci* sequence

$$\text{fib} = 0100101001001 \cdots$$

is defined by $\text{fib} \; = \; f^{\infty}(0)$ for $f(0) = 01$, $f(1) = 0$

These are pure morphic

For $f(0) = 0$, $f(1) = 10$, $f(2) = 210$, $\tau(0) = 0$, $\tau(1) = \tau(2) = 1$
we obtain

$$\tau(f^{\infty}(2)) \; = \; 11010010^3 10^4 10^5 1 \cdots$$

It is morphic; it is easily shown not to be pure morphic

This talk

This talk

In this talk we will focus on three aspects of morphic sequences:

This talk

In this talk we will focus on three aspects of morphic sequences:

- Equivalent characterizations of the class of morphic sequences and the relation to numeration systems

This talk

In this talk we will focus on three aspects of morphic sequences:

- Equivalent characterizations of the class of morphic sequences and the relation to numeration systems
- Visualization by turtle graphics

This talk

In this talk we will focus on three aspects of morphic sequences:

- Equivalent characterizations of the class of morphic sequences and the relation to numeration systems
- Visualization by turtle graphics
- How to prove that two representations give the same sequence

The class of morphic sequences

The class of morphic sequences

The class of morphic sequences is closed under several operations, like

- adding or removing a string at the front,
- applying morphisms,
- take arithmetic subsequence like even

The class of morphic sequences

The class of morphic sequences is closed under several operations, like

- adding or removing a string at the front,
- applying morphisms,
- take arithmetic subsequence like even

Just like the class of *regular languages* is closed under several operations and has several equivalent characterizations, one may expect that a similar robustness of the class of morphic sequences gives rise to equivalent characterizations

The class of morphic sequences

The class of morphic sequences is closed under several operations, like

- adding or removing a string at the front,
- applying morphisms,
- take arithmetic subsequence like even

Just like the class of *regular languages* is closed under several operations and has several equivalent characterizations, one may expect that a similar robustness of the class of morphic sequences gives rise to equivalent characterizations

We investigate some results in this direction, along the lines of similar characterizations of *automatic sequences*, being morphic sequences for which all $f(b)$ have the same length

Tree structure

Tree structure

The recursive calls in the definition of morphic sequence give rise to a *tree structure*

Tree structure

The recursive calls in the definition of morphic sequence give rise to a *tree structure*

For the Fibonacci sequence fib this starts in

$$a = 0$$

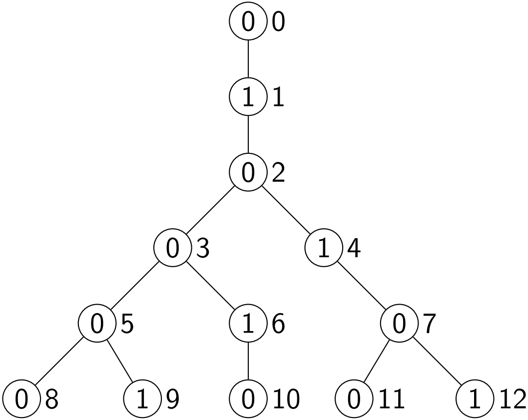
$$u = 1$$

$$f(u) = 0$$

$$f^2(u) = 01$$

$$f^3(u) = 010$$

$$f^4(u) = 01001$$



Every internal node labeled by a has $|f(a)|$ children, so Thue-Morse $\mathbf{t}$ gives a binary tree

Every internal node labeled by a has $|f(a)|$ children, so Thue-Morse $\mathbf{t}$ gives a binary tree

Every such f gives rise to a *numeration system*: numbering the nodes in a breadth-first way as we did gives rise to a bijection between $\mathbb{N}$ and the set of finite paths in the tree

Every internal node labeled by a has $|f(a)|$ children, so Thue-Morse $\mathbf{t}$ gives a binary tree

Every such f gives rise to a *numeration system*: numbering the nodes in a breadth-first way as we did gives rise to a bijection between $\mathbb{N}$ and the set of finite paths in the tree

The tree is *rational*, that is, has only finitely many distinct subtrees: one for every alphabet symbol

Every internal node labeled by a has $|f(a)|$ children, so Thue-Morse $\mathbf{t}$ gives a binary tree

Every such f gives rise to a *numeration system*: numbering the nodes in a breadth-first way as we did gives rise to a bijection between $\mathbb{N}$ and the set of finite paths in the tree

The tree is *rational*, that is, has only finitely many distinct subtrees: one for every alphabet symbol

Sharing all equal subtrees gives rise to a finite representation, a *mix-DFAO*

Every internal node labeled by a has $|f(a)|$ children, so Thue-Morse $\mathbf{t}$ gives a binary tree

Every such f gives rise to a *numeration system*: numbering the nodes in a breadth-first way as we did gives rise to a bijection between $\mathbb{N}$ and the set of finite paths in the tree

The tree is *rational*, that is, has only finitely many distinct subtrees: one for every alphabet symbol

Sharing all equal subtrees gives rise to a finite representation, a *mix-DFAO*

DFAO is DFA with *output*, that is, the symbol to be produced

Every internal node labeled by a has $|f(a)|$ children, so Thue-Morse $\mathbf{t}$ gives a binary tree

Every such f gives rise to a *numeration system*: numbering the nodes in a breadth-first way as we did gives rise to a bijection between $\mathbb{N}$ and the set of finite paths in the tree

The tree is *rational*, that is, has only finitely many distinct subtrees: one for every alphabet symbol

Sharing all equal subtrees gives rise to a finite representation, a *mix-DFAO*

DFAO is DFA with *output*, that is, the symbol to be produced

Every node $=$ state corresponds to a symbol a and has outgoing arrows labeled by $0, 1, \ldots, m-1$ where $m = |f(a)|$

Every internal node labeled by a has $|f(a)|$ children, so Thue-Morse $\mathbf{t}$ gives a binary tree

Every such f gives rise to a *numeration system*: numbering the nodes in a breadth-first way as we did gives rise to a bijection between $\mathbb{N}$ and the set of finite paths in the tree

The tree is *rational*, that is, has only finitely many distinct subtrees: one for every alphabet symbol

Sharing all equal subtrees gives rise to a finite representation, a *mix-DFAO*

DFAO is DFA with *output*, that is, the symbol to be produced

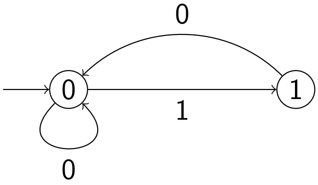
Every node = state corresponds to a symbol a and has outgoing arrows labeled by $0, 1, \ldots, m-1$ where $m = |f(a)|$

This mix-DFAO can be seen as a DFAO in which the transition function δ is *partial*

The alphabet is assumed to be of the shape $\{0, 1, \ldots, m-1\}$

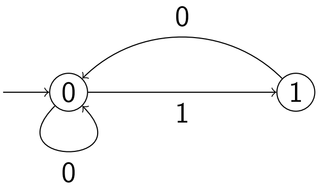
The alphabet is assumed to be of the shape $\{0, 1, \ldots, m-1\}$

For fib the mix-DFAO reads



The alphabet is assumed to be of the shape $\{0, 1, \ldots, m-1\}$

For fib the mix-DFAO reads



Theorem

A sequence is morphic if and only if it is represented by a mix-DFAO

Ignoring the output, such a mix-DFA gives rise to a *numeration system*

Ignoring the output, such a mix-DFA gives rise to a *numeration system*

Now we present such numeration systems in a much more general setting along the lines of the books *Formal Languages, Automata and Numeration Systems* by Michel Rigo

Ignoring the output, such a mix-DFA gives rise to a *numeration system*

Now we present such numeration systems in a much more general setting along the lines of the books *Formal Languages, Automata and Numeration Systems* by Michel Rigo

An *abstract numeration system (ANS)* is a regular language L over the alphabet $\{0, 1, \ldots, m-1\}$

Ignoring the output, such a mix-DFA gives rise to a *numeration system*

Now we present such numeration systems in a much more general setting along the lines of the books *Formal Languages, Automata and Numeration Systems* by Michel Rigo

An *abstract numeration system (ANS)* is a regular language L over the alphabet $\{0, 1, \ldots, m-1\}$

It defines a representation function $\text{rep}_L : \mathbb{N} \to L$, being bijective and monotone wrt the genealogical order on L, that is, first look at the length, and then compare words of the same length lexicographically

Ignoring the output, such a mix-DFA gives rise to a *numeration system*

Now we present such numeration systems in a much more general setting along the lines of the books *Formal Languages, Automata and Numeration Systems* by Michel Rigo

An *abstract numeration system (ANS)* is a regular language L over the alphabet $\{0, 1, \ldots, m-1\}$

It defines a representation function $\text{rep}_L : \mathbb{N} \to L$, being bijective and monotone wrt the genealogical order on L, that is, first look at the length, and then compare words of the same length lexicographically

If L consists of the words not starting in 0 then this corresponds to the normal m-ary representation

For such an ANS L a sequence σ is called *L-automatic* if there exists a partial DFAO such that

$$\sigma(i) \ = \ \mu(\delta(q_0, \mathrm{rep}_L(i)))$$

for all $i \in \mathbb{N}$, where μ, δ, q_0 are the output function, transition function and initial state of the DFAO

For such an ANS L a sequence σ is called *L-automatic* if there exists a partial DFAO such that

$$\sigma(i) \; = \; \mu(\delta(q_0, \mathrm{rep}_L(i)))$$

for all $i \in \mathbb{N}$, where μ, δ, q_0 are the output function, transition function and initial state of the DFAO

Theorem

A sequence is morphic if and only if it is L-automatic for some ANS L

For such an ANS L a sequence σ is called *L-automatic* if there exists a partial DFAO such that

$$\sigma(i) \;=\; \mu(\delta(q_0, \mathrm{rep}_L(i)))$$

for all $i \in \mathbb{N}$, where μ, δ, q_0 are the output function, transition function and initial state of the DFAO

Theorem

A sequence is morphic if and only if it is L-automatic for some ANS L

Here L-automatic allows much more freedom than the mix-DFAO representation we gave earlier

For such an ANS L a sequence σ is called *L-automatic* if there exists a partial DFAO such that

$$\sigma(i) \; = \; \mu(\delta(q_0, \mathrm{rep}_L(i)))$$

for all $i \in \mathbb{N}$, where μ, δ, q_0 are the output function, transition function and initial state of the DFAO

Theorem

A sequence is morphic if and only if it is L-automatic for some ANS L

Here L-automatic allows much more freedom than the mix-DFAO representation we gave earlier

Both theorems are correct, in fact the proof that any morphic sequence is L-automatic in Rigo's book essentially uses the mix-DFAO representation as we did in our proof

The terminology *mix-DFAO* was introduced in the LATA2013 paper by Endrullis, Grabmayer and Hendriks

The terminology *mix-DFAO* was introduced in the LATA2013 paper by Endrullis, Grabmayer and Hendriks

There it was used to define the *mix-automatic sequences* in which the sequence defined by a mix-DFAO is different: to compute $\sigma(i)$ the sequence $\text{rep}(i)$ is entered to the mix-DFAO in reverse order

The terminology *mix-DFAO* was introduced in the LATA2013 paper by Endrullis, Grabmayer and Hendriks

There it was used to define the *mix-automatic sequences* in which the sequence defined by a mix-DFAO is different: to compute $\sigma(i)$ the sequence $\mathrm{rep}(i)$ is entered to the mix-DFAO in reverse order

Their main result is that the classes of morphic sequences and mix-automatic sequences are incomparable

One more characterization of morphic sequences

Numbering the nodes of a tree by natural numbers yields a *parent* function $P : \mathbb{N}_{>0} \to \mathbb{N}$

One more characterization of morphic sequences

Numbering the nodes of a tree by natural numbers yields a *parent* function $P : \mathbb{N}_{>0} \to \mathbb{N}$

If the tree is rational, the corresponding function P is called a *rational tree function*

One more characterization of morphic sequences

Numbering the nodes of a tree by natural numbers yields a *parent* function $P : \mathbb{N}_{>0} \to \mathbb{N}$

If the tree is rational, the corresponding function P is called a *rational tree function*

For a sequence σ, a rational tree function P and a number n let $\sigma[n]$ be the subsequence of σ obtained by only keeping the elements of σ on positions k for which $P^m(k) = n$ for some m

Theorem

A sequence σ over Σ is morphic if and only if a rational tree function $P : \mathbb{N}_{>0} \to \mathbb{N}$ exists such that the set

$$\{\sigma[n] \mid n \in \mathbb{N}\}$$

of subsequences of σ is finite.

Conclusions characterizations of morphic sequences

Conclusions characterizations of morphic sequences

- Automatic sequences have several equivalent characterizations, based on automata (DFAO), morphic sequences and finiteness of kernel

Conclusions characterizations of morphic sequences

- Automatic sequences have several equivalent characterizations, based on automata (DFAO), morphic sequences and finiteness of kernel
- For morphic sequences we also gave a characterization by automata, essentially by DFAOs for which the transition function is *partial*

Conclusions characterizations of morphic sequences

- Automatic sequences have several equivalent characterizations, based on automata (DFAO), morphic sequences and finiteness of kernel
- For morphic sequences we also gave a characterization by automata, essentially by DFAOs for which the transition function is *partial*
- The characterization of automatic sequences by finiteness of the *kernel* is essentially about finiteness of a class of subsequences, we gave a similar characterization for morphic sequences

Conclusions characterizations of morphic sequences

- Automatic sequences have several equivalent characterizations, based on automata (DFAO), morphic sequences and finiteness of kernel
- For morphic sequences we also gave a characterization by automata, essentially by DFAOs for which the transition function is *partial*
- The characterization of automatic sequences by finiteness of the *kernel* is essentially about finiteness of a class of subsequences, we gave a similar characterization for morphic sequences
- Feeding number representations in reverse direction into DFAO yields the same class of automatic sequences, for the variant for morphic sequences this is not the case

Turtle figures

Turtle figures

One reason to consider morphic seqeunces is that they give rise to amazing *turtle figures*

Turtle figures

One reason to consider morphic seqeunces is that they give rise to amazing *turtle figures*

For every $a \in A$ choose an angle $\alpha(a) \in \mathbf{R}$

Turtle figures

One reason to consider morphic seqeunces is that they give rise to amazing *turtle figures*

For every $a \in A$ choose an angle $\alpha(a) \in \mathbf{R}$

Then a sequence σ over A has a *turtle curve*:

Turtle figures

One reason to consider morphic seqeunces is that they give rise to amazing *turtle figures*

For every $a \in A$ choose an angle $\alpha(a) \in \mathbf{R}$

Then a sequence σ over A has a *turtle curve*:

Start in $(0, 0)$ and draw a segment of unit length in the direction $\alpha(\sigma(0))$, by which the current direction is $\alpha(\sigma(0))$

Turtle figures

One reason to consider morphic seqeunces is that they give rise to amazing *turtle figures*

For every $a \in A$ choose an angle $\alpha(a) \in \mathbf{R}$

Then a sequence σ over A has a *turtle curve*:

Start in $(0,0)$ and draw a segment of unit length in the direction $\alpha(\sigma(0))$, by which the current direction is $\alpha(\sigma(0))$

Next for $i = 1, 2, 3, \ldots$ continue by adding $\alpha(\sigma(i))$ to the current direction and draw a segment in this direction

Turtle figures

One reason to consider morphic seqeunces is that they give rise to amazing *turtle figures*

For every $a \in A$ choose an angle $\alpha(a) \in \mathbf{R}$

Then a sequence σ over A has a *turtle curve*:

Start in $(0, 0)$ and draw a segment of unit length in the direction $\alpha(\sigma(0))$, by which the current direction is $\alpha(\sigma(0))$

Next for $i = 1, 2, 3, \ldots$ continue by adding $\alpha(\sigma(i))$ to the current direction and draw a segment in this direction

The *turtle figure* is defined to be the union of all resulting segments

As a first example, consider $f(0) = 0$, $f(1) = 10$, $f(2) = 210$, giving

$$f^\infty(2) \ = \ 21010010^3 10^4 10^5 1 \cdots$$

As a first example, consider $f(0) = 0$, $f(1) = 10$, $f(2) = 210$, giving

$$f^\infty(2) = 21010010^3 10^4 10^5 1 \cdots$$

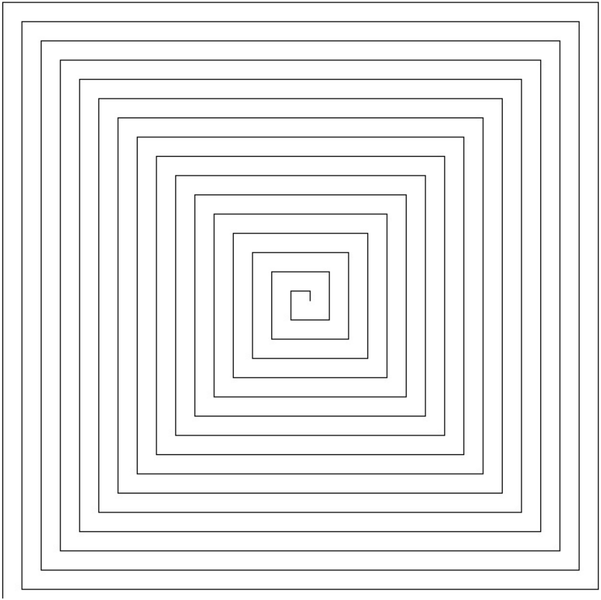
Choose $\alpha(0) = 0$, $\alpha(1) = \alpha(2) = 90^\circ$

As a first example, consider $f(0) = 0$, $f(1) = 10$, $f(2) = 210$, giving

$$f^{\infty}(2) \; = \; 21010010^3 10^4 10^5 1 \cdots$$

Choose $\alpha(0) = 0$, $\alpha(1) = \alpha(2) = 90^\circ$

This gives rise to the following turtle figure



For a fixed number n every pure morphic sequence $f^\infty(a)$ is composed from $f^n(b)$, for b running over the alphabet

For a fixed number n every pure morphic sequence $f^\infty(a)$ is composed from $f^n(b)$, for b running over the alphabet

For instance, the *Thue-Morse* sequence

$$\mathbf{t} = 0110100110010110\cdots$$

defined by $\mathbf{t} = f^\infty(0)$ for $f(0) = 01$, $f(1) = 10$ is composed from $f^3(0) = 01101001$ and $f^3(1) = 10010110$

For a fixed number n every pure morphic sequence $f^\infty(a)$ is composed from $f^n(b)$, for b running over the alphabet

For instance, the *Thue-Morse* sequence

$$\mathbf{t} = 0110100110010110\cdots$$

defined by $\mathbf{t} = f^\infty(0)$ for $f(0) = 01$, $f(1) = 10$ is composed from $f^3(0) = 01101001$ and $f^3(1) = 10010110$

One proves that if $2^n(\alpha(0) + \alpha(1))$ is a multiple of $360^\circ = 2\pi$, then both $f^{n+2}(0)$ and $f^{n+2}(1)$ give rise to turtle figures that end where they started, both in position and angle

For a fixed number n every pure morphic sequence $f^\infty(a)$ is composed from $f^n(b)$, for b running over the alphabet

For instance, the *Thue-Morse* sequence

$$\mathbf{t} = 0110100110010110\cdots$$

defined by $\mathbf{t} = f^\infty(0)$ for $f(0) = 01$, $f(1) = 10$ is composed from $f^3(0) = 01101001$ and $f^3(1) = 10010110$

One proves that if $2^n(\alpha(0) + \alpha(1))$ is a multiple of $360^\circ = 2\pi$, then both $f^{n+2}(0)$ and $f^{n+2}(1)$ give rise to turtle figures that end where they started, both in position and angle

Hence in that case the turtle figure of the infinite sequence $\mathbf{t} = f^\infty(0)$ draws these two finite turtle figures over and over again, so is *finite*

For a fixed number n every pure morphic sequence $f^\infty(a)$ is composed from $f^n(b)$, for b running over the alphabet

For instance, the *Thue-Morse* sequence

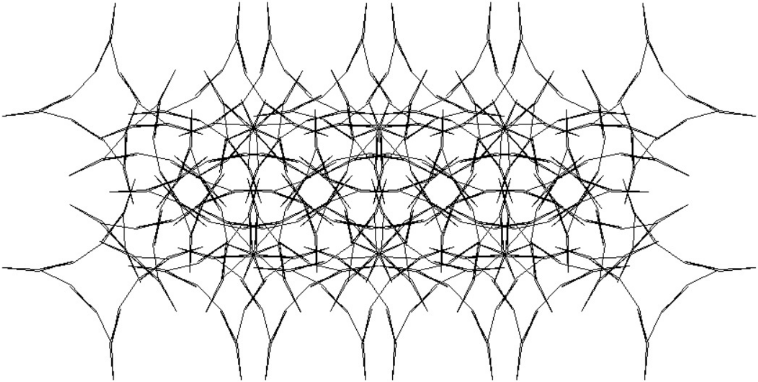
$$\mathbf{t} = 0110100110010110\cdots$$

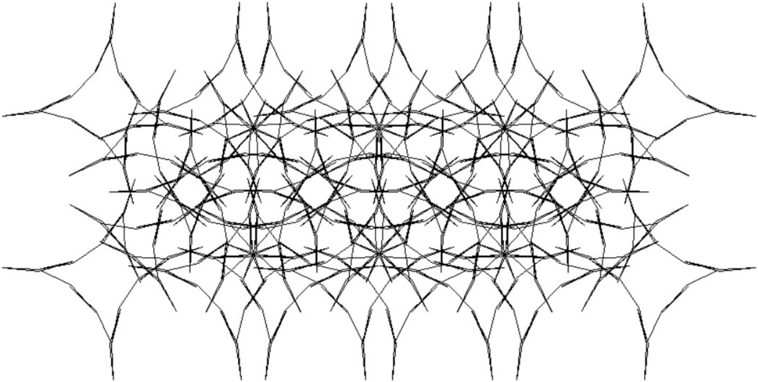
defined by $\mathbf{t} = f^\infty(0)$ for $f(0) = 01$, $f(1) = 10$ is composed from $f^3(0) = 01101001$ and $f^3(1) = 10010110$

One proves that if $2^n(\alpha(0) + \alpha(1))$ is a multiple of $360^\circ = 2\pi$, then both $f^{n+2}(0)$ and $f^{n+2}(1)$ give rise to turtle figures that end where they started, both in position and angle

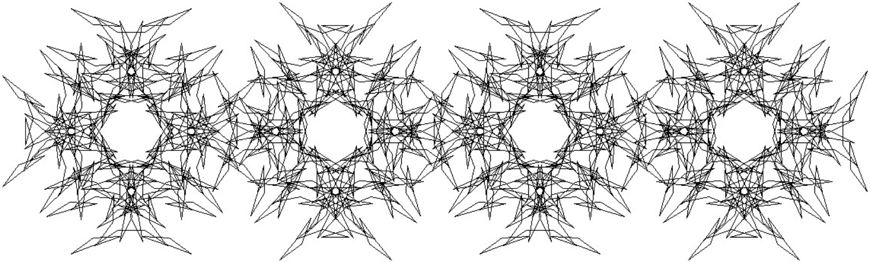
Hence in that case the turtle figure of the infinite sequence $\mathbf{t} = f^\infty(0)$ draws these two finite turtle figures over and over again, so is *finite*

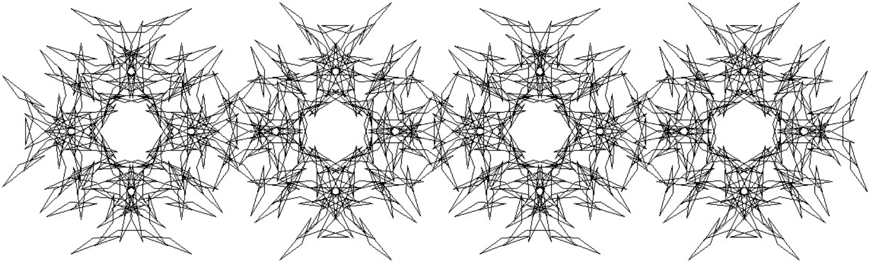
We give a few examples of resulting turtle figures of $\mathbf{t}$ where $2^n(\alpha(0) + \alpha(1))$ is a multiple of 360°



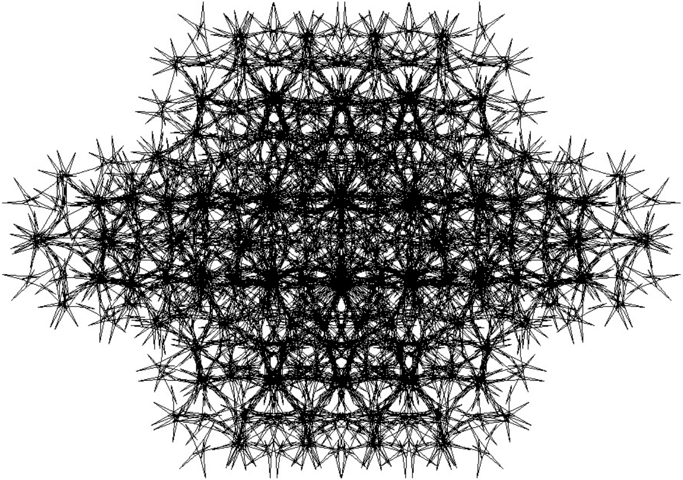


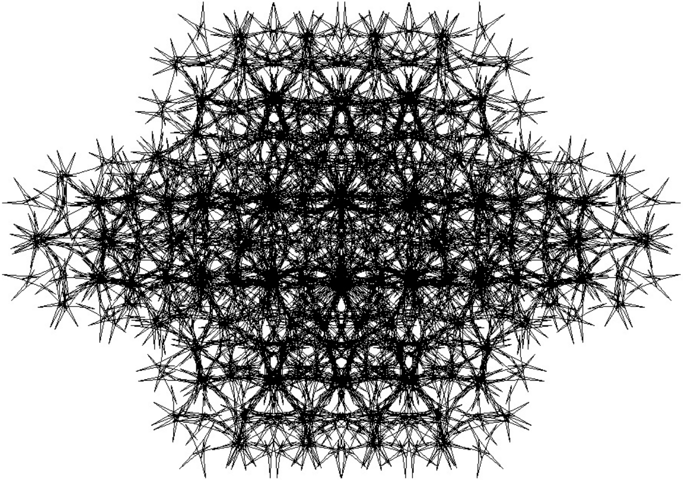
$f(0) = 01$, $f(1) = 10$, $\alpha(0) = \frac{\pi}{8}$, $\alpha(1) = \frac{63\pi}{64}$





$f(0) = 01,\ f(1) = 10,\ \alpha(0) = \frac{3\pi}{16},\ \alpha(1) = \frac{117\pi}{128}$





$f(0) = 01$, $f(1) = 10$, $\alpha(0) = \frac{61\pi}{64}$, $\alpha(1) = \frac{33\pi}{1024}$

More finite turtle figures

More finite turtle figures

Over the alphabet $\{0, 1\}$ for every f the sequence $f^{\infty}(0)$ is composed from $f^n(0)$ and $f^n(1)$, for any fixed n

More finite turtle figures

Over the alphabet $\{0,1\}$ for every f the sequence $f^\infty(0)$ is composed from $f^n(0)$ and $f^n(1)$, for any fixed n

If the turtle figure of $f^n(0)$ ends in a rational angle different from the initial angle, then the turtle figure of the periodic sequence $(f^n(0))^\infty$ is finite

More finite turtle figures

Over the alphabet $\{0, 1\}$ for every f the sequence $f^\infty(0)$ is composed from $f^n(0)$ and $f^n(1)$, for any fixed n

If the turtle figure of $f^n(0)$ ends in a rational angle different from the initial angle, then the turtle figure of the periodic sequence $(f^n(0))^\infty$ is finite

If moreover the turtle figure of $f^n(1)$ ends in its initial position and angle, then the turtle curve of the sequence $f^\infty(0)$, being composed from $f^n(0)$ and $f^n(1)$ will be finite

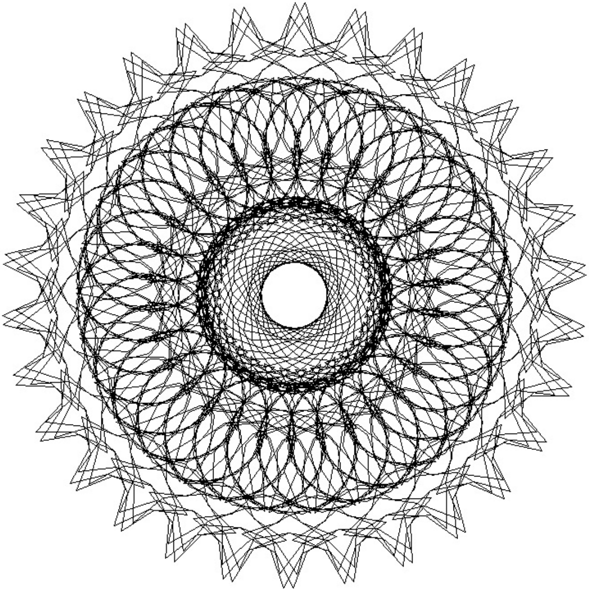
More finite turtle figures

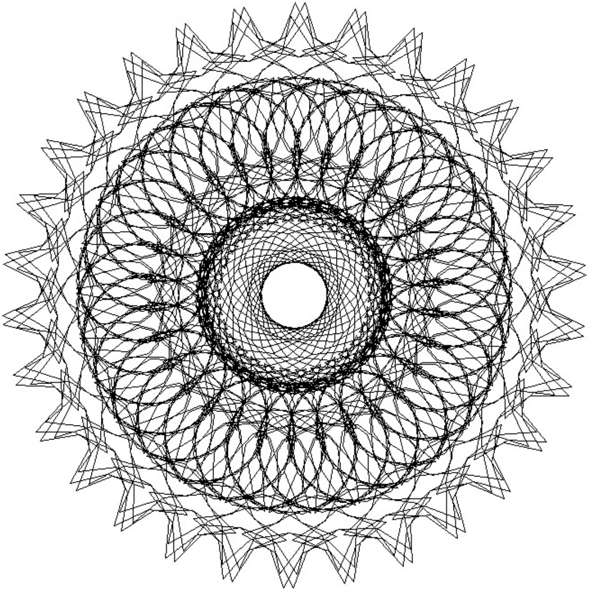
Over the alphabet $\{0,1\}$ for every f the sequence $f^\infty(0)$ is composed from $f^n(0)$ and $f^n(1)$, for any fixed n

If the turtle figure of $f^n(0)$ ends in a rational angle different from the initial angle, then the turtle figure of the periodic sequence $(f^n(0))^\infty$ is finite

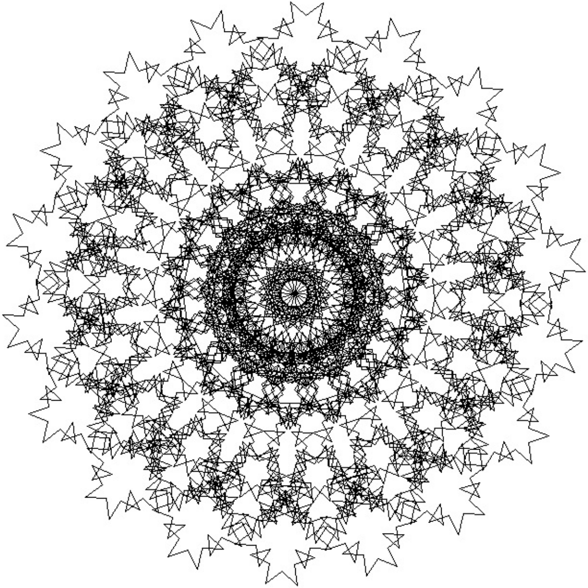
If moreover the turtle figure of $f^n(1)$ ends in its initial position and angle, then the turtle curve of the sequence $f^\infty(0)$, being composed from $f^n(0)$ and $f^n(1)$ will be finite

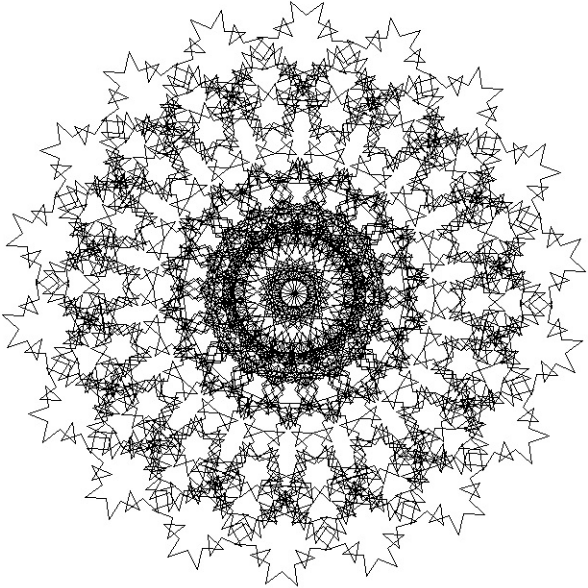
We will give a few examples of this



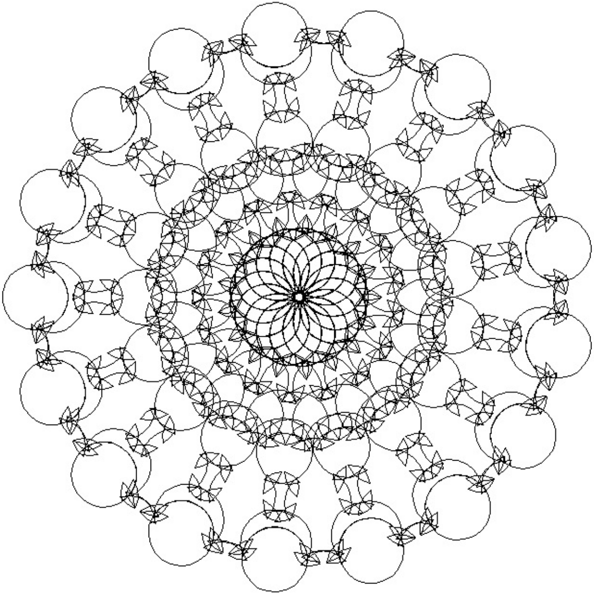


$f(0) = 0101$, $f(1) = 11$, $\alpha(0) = -132^\circ$, $\alpha(1) = 33\frac{3}{4}^\circ = \frac{3\pi}{16}$





$f(0) = 01,\ f(1) = 00,\ \alpha(0) = 140^\circ,\ \alpha(1) = -80^\circ$



Fractal turtle figures

Fractal turtle figures

Apart from all these finite turtle figures, also infinite turtle figures are of interest

Fractal turtle figures

Apart from all these finite turtle figures, also infinite turtle figures are of interest

In particular *fractal turtle figures*, in its simplest form turtle figures of which the set P of end points of all the (infinitely many) end points of the segments have the following fractal property:

$$cP \subseteq P$$

for some magnifying factor $c > 1$, where the points in P are considered to be vectors with respect to some origin

Fractal turtle figures

Apart from all these finite turtle figures, also infinite turtle figures are of interest

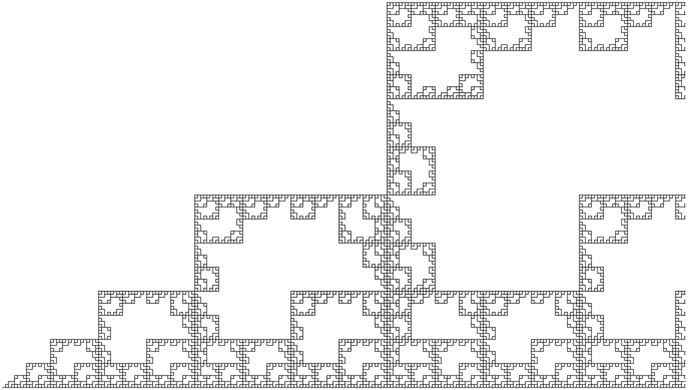
In particular *fractal turtle figures*, in its simplest form turtle figures of which the set P of end points of all the (infinitely many) end points of the segments have the following fractal property:

$$cP \subseteq P$$

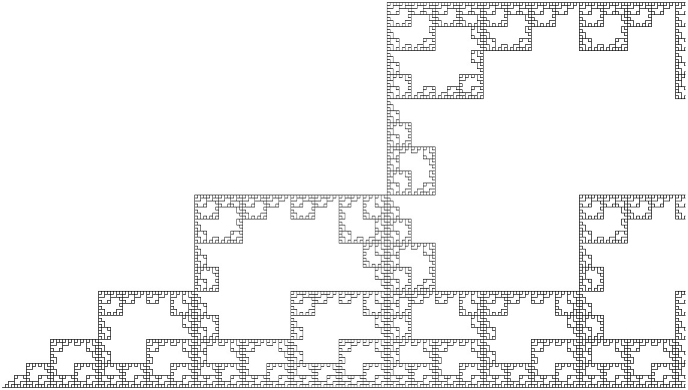
for some magnifying factor $c > 1$, where the points in P are considered to be vectors with respect to some origin

An immediate consequence of this definition is that every fractal turtle figure is infinite

Example

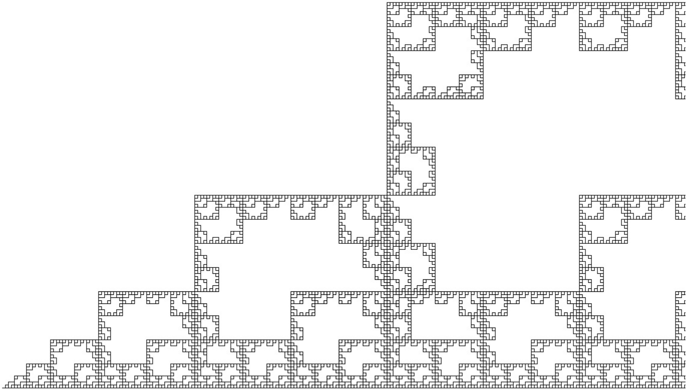


Example



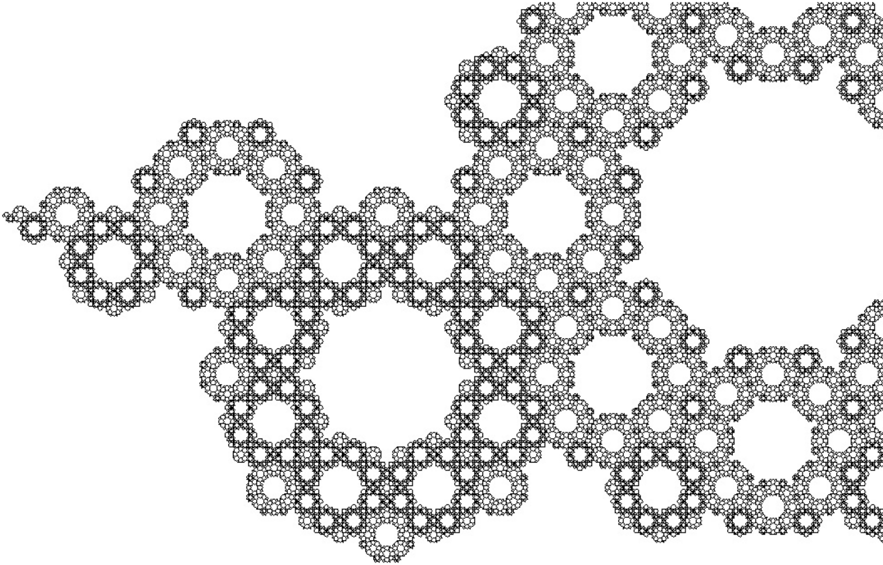
obtained as the turtle figure of $f^{\infty}(0)$ for $f(0) = 001111$, $f(1) = 10$, $\alpha(0) = 0$, $\alpha(1) = 90^{\circ}$, giving a magnifying factor $c = 2$

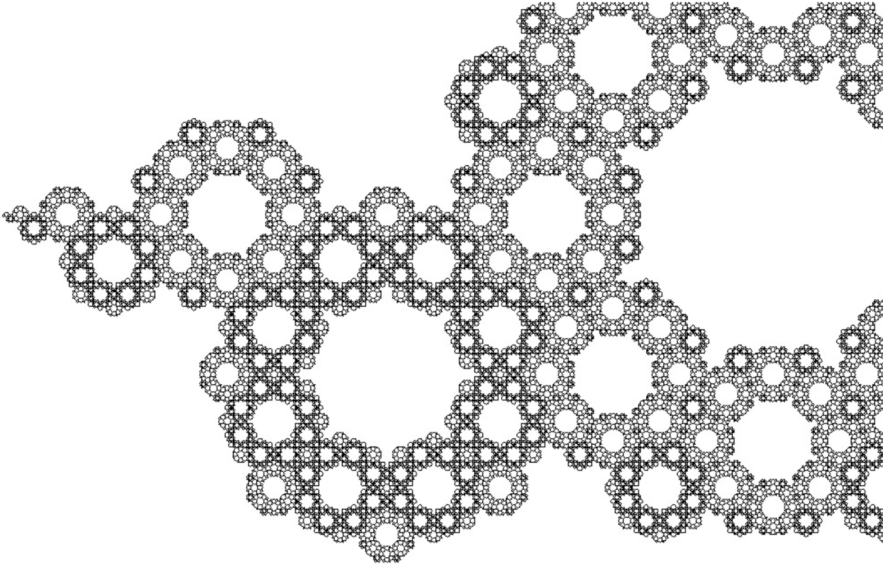
Example



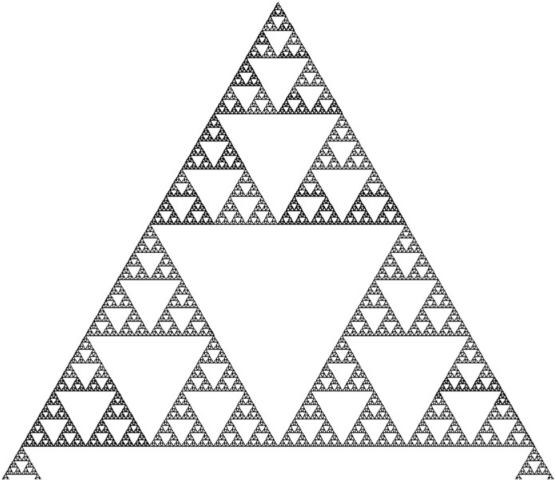
obtained as the turtle figure of $f^\infty(0)$ for $f(0) = 001111$, $f(1) = 10$, $\alpha(0) = 0$, $\alpha(1) = 90^\circ$, giving a magnifying factor $c = 2$

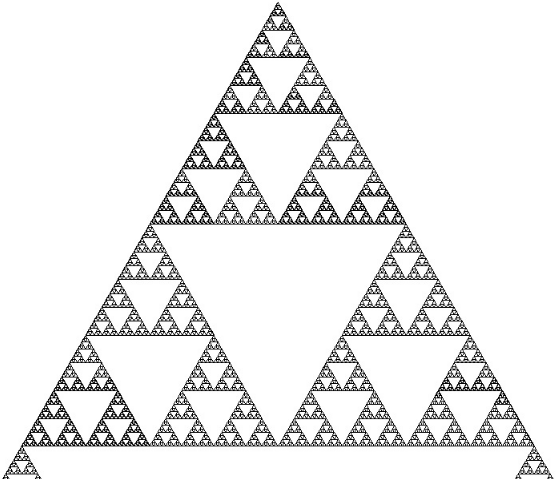
Key idea: applying f causes scaling up factor c in turtle figure





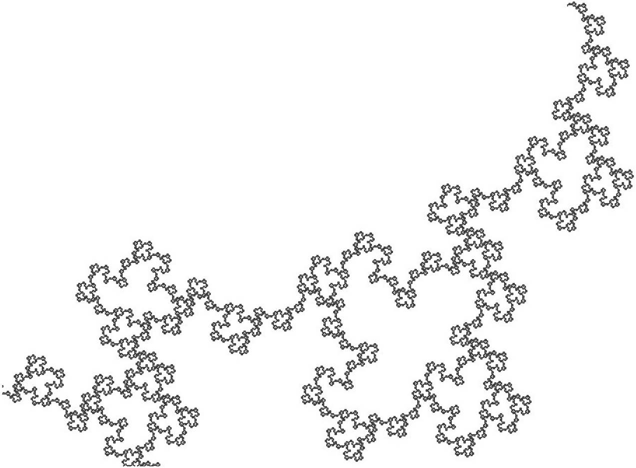
$f(0) = 011111$, $f(1) = 00$, $\alpha(0) = 45^\circ$, $\alpha(1) = -90^\circ$



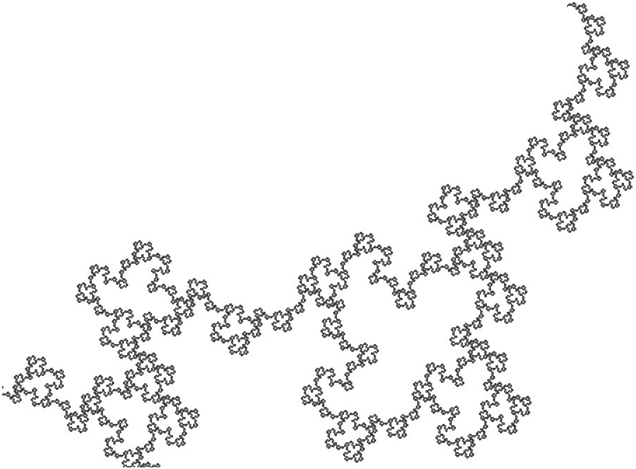


Sierpinsky triangle, obtained by $f(0) = 00001$, $f(1) = 11$, $\alpha(0) = 120^\circ$, $\alpha(1) = 0$

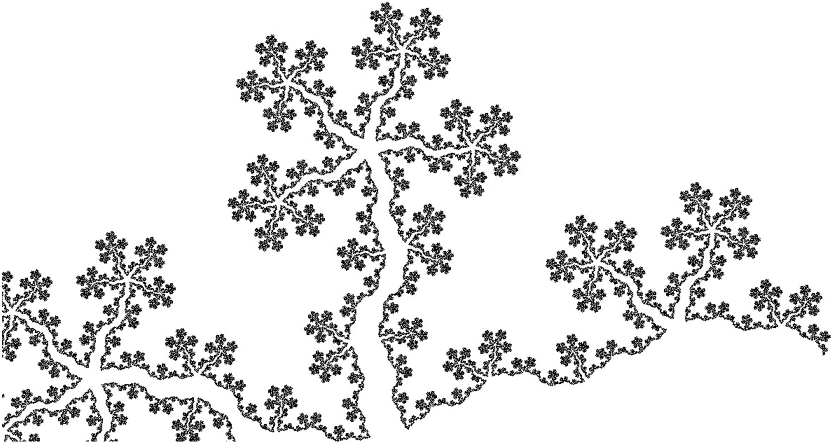
More complicated criteria yield fractal turtle figures with *rotation*

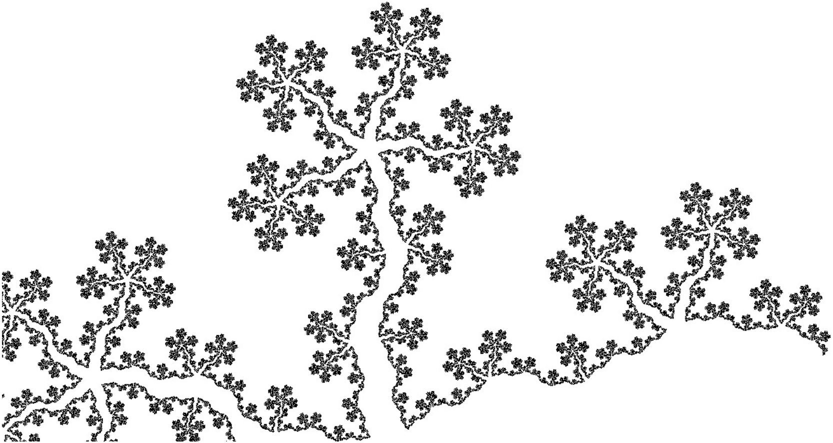


More complicated criteria yield fractal turtle figures with *rotation*



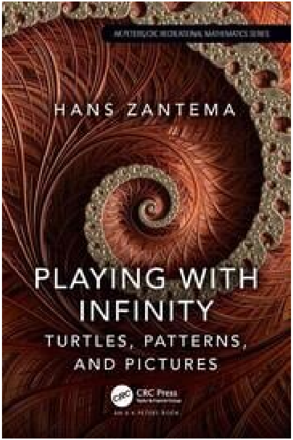
$f(0) = 0101111$, $f(1) = 110$, $\alpha(0) = 90^\circ$, $\alpha(1) = -90^\circ$



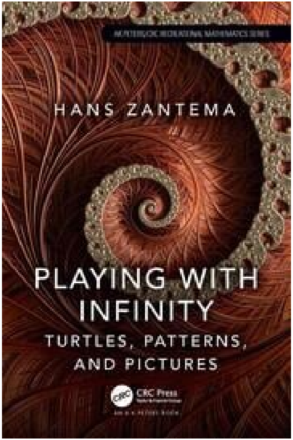


$f(0) = 000110$, $f(1) = 100110$, $\alpha(0) = 70^\circ$, $\alpha(1) = -105^\circ$

All these examples and underlying theory are presented in



All these examples and underlying theory are presented in



This book is written for a wide audience, and apart from turtle graphics of morphic sequences it contains a general mathematical introduction to infinity, and many mathematical challenges

The last topic in this presentation is on automatically *proving equality* of morphic sequences (Z24)

The last topic in this presentation is on automatically *proving equality* of morphic sequences (Z24)

Example:

The last topic in this presentation is on automatically *proving equality* of morphic sequences (Z24)

Example:

It happens that fib is equal to $\rho(g^\infty(0))$ for g, ρ defined by

$$g(0) = 02, g(1) = 021, g(2) = 102, \rho(0) = \rho(1) = 0, \rho(2) = 1$$

The last topic in this presentation is on automatically *proving equality* of morphic sequences (Z24)

Example:

It happens that fib is equal to $\rho(g^\infty(0))$ for g, ρ defined by

$$g(0) = 02, g(1) = 021, g(2) = 102, \rho(0) = \rho(1) = 0, \rho(2) = 1$$

How to prove this?

The last topic in this presentation is on automatically *proving equality* of morphic sequences (Z24)

Example:

It happens that fib is equal to $\rho(g^\infty(0))$ for g, ρ defined by

$$g(0) = 02, g(1) = 021, g(2) = 102, \rho(0) = \rho(1) = 0, \rho(2) = 1$$

How to prove this?

fib $= f^\infty(0)$ for $f(0) = 01, f(1) = 0$, also if f is replaced by f^2:

$$f(0) = 010, \; f(1) = 01$$

The last topic in this presentation is on automatically *proving equality* of morphic sequences (Z24)

Example:

It happens that fib is equal to $\rho(g^\infty(0))$ for g, ρ defined by

$$g(0) = 02, g(1) = 021, g(2) = 102, \rho(0) = \rho(1) = 0, \rho(2) = 1$$

How to prove this?

fib $= f^\infty(0)$ for $f(0) = 01, f(1) = 0$, also if f is replaced by f^2:

$$f(0) = 010,\ f(1) = 01$$

Claim to be proved: $f^\infty(0) = \rho(g^\infty(0))$

We will prove the following properties simultaneously by induction on n:
(0) $f^{n-1}(01) = \rho(g^n(0))$
(1) $f^{n-1}(010) = \rho(g^n(1))$
(2) $f^{n-1}(001) = \rho(g^n(2))$

We will prove the following properties simultaneously by induction on n:
(0) $f^{n-1}(01) = \rho(g^n(0))$
(1) $f^{n-1}(010) = \rho(g^n(1))$
(2) $f^{n-1}(001) = \rho(g^n(2))$

Then our claim follows from (0)

We will prove the following properties simultaneously by induction on n:
(0) $f^{n-1}(01) = \rho(g^n(0))$
(1) $f^{n-1}(010) = \rho(g^n(1))$
(2) $f^{n-1}(001) = \rho(g^n(2))$

Then our claim follows from (0)

Basis $n = 1$ of induction:
$f^0(01) = 01 = \rho(g(0))$
$f^0(010) = 010 = \rho(g(1))$
$f^0(001)) = 001 = \rho(g(2))$

We will prove the following properties simultaneously by induction on n:
(0) $f^{n-1}(01) = \rho(g^n(0))$
(1) $f^{n-1}(010) = \rho(g^n(1))$
(2) $f^{n-1}(001) = \rho(g^n(2))$

Then our claim follows from (0)

Basis $n = 1$ of induction:
$f^0(01) = 01 = \rho(g(0))$
$f^0(010) = 010 = \rho(g(1))$
$f^0(001)) = 001 = \rho(g(2))$

Hence basis of induction proved

Induction step part (0):

Induction step part (0):

$$f^n(01)) = f^{n-1}(f(01)) = f^{n-1}(010\,01)$$

Induction step part (0):

$$f^n(01)) = f^{n-1}(f(01)) = f^{n-1}(010\,01)$$

$$= f^{n-1}(01\,001) = \rho(g^n(02)) = \rho(g^{n+1}(0))$$

using IH (0) and IH (2)

Induction step part (0):

$$f^n(01)) = f^{n-1}(f(01)) = f^{n-1}(010\,01)$$

$$= f^{n-1}(01\,001) = \rho(g^n(02)) = \rho(g^{n+1}(0))$$

using IH (0) and IH (2)

proving part (0)

Induction step part (1):

$$f^n(010) = f^{n-1}(01001010) = \rho(g^n(021)) = \rho(g^{n+1}(1))$$

using IH (0) and IH (2) and IH (1)

Induction step part (1):

$$f^n(010) = f^{n-1}(01001010) = \rho(g^n(021)) = \rho(g^{n+1}(1))$$

using IH (0) and IH (2) and IH (1)

Induction step part (2):

$$f^n(001) = f^{n-1}(01001001) = \rho(g^n(102)) = \rho(g^{n+1}(2))$$

using IH (1) and IH (0) and IH (2)

Induction step part (1):

$$f^n(010) = f^{n-1}(01001010) = \rho(g^n(021)) = \rho(g^{n+1}(1))$$

using IH (0) and IH (2) and IH (1)

Induction step part (2):

$$f^n(001) = f^{n-1}(01001001) = \rho(g^n(102)) = \rho(g^{n+1}(2))$$

using IH (1) and IH (0) and IH (2)

Induction step proved, hence claim proved

You might think that I found this proof and typed it myself

You might think that I found this proof and typed it myself

That's not the case: I wrote a prototype tool that searches for a general pattern, and automatically generates the proof as we just gave it

You might think that I found this proof and typed it myself

That's not the case: I wrote a prototype tool that searches for a general pattern, and automatically generates the proof as we just gave it

The general pattern is given by the following theorem in which the alphabet for g is $\{0, 1, \ldots, n\}$

You might think that I found this proof and typed it myself

That's not the case: I wrote a prototype tool that searches for a general pattern, and automatically generates the proof as we just gave it

The general pattern is given by the following theorem in which the alphabet for g is $\{0, 1, \ldots, n\}$

Theorem

For $i = 0, 1, \ldots, n$ let w_i be the prefix in front of the first occurrence of i in $g^\infty(0)$, and write $u_i = f^\infty(0)_{|g(w_i)|,|g(w_i i)|}$
For $i = 0, 1, \ldots, n$ assume that $\tau(u_i) = \rho(g(i))$ and $f(u_i) = u_{a_0} \cdots u_{a_{k-1}}$ for $g(i) = a_0 \cdots a_{k-1}$

You might think that I found this proof and typed it myself

That's not the case: I wrote a prototype tool that searches for a general pattern, and automatically generates the proof as we just gave it

The general pattern is given by the following theorem in which the alphabet for g is $\{0, 1, \ldots, n\}$

Theorem

For $i = 0, 1, \ldots, n$ let w_i be the prefix in front of the first occurrence of i in $g^\infty(0)$, and write $u_i = f^\infty(0)_{|g(w_i)|,|g(w_i i)|}$
For $i = 0, 1, \ldots, n$ assume that $\tau(u_i) = \rho(g(i))$ and $f(u_i) = u_{a_0} \cdots u_{a_{k-1}}$ for $g(i) = a_0 \cdots a_{k-1}$

Then $\tau(f^\infty(0)) = \rho(g^\infty(0))$

The proof requires $\tau(f^n(u_i)) = \rho(g^{n+1}(0))$ for all n which is only possible if f and g have the same *dominant eigenvalue*

The proof requires $\tau(f^n(u_i)) = \rho(g^{n+1}(0))$ for all n which is only possible if f and g have the same *dominant eigenvalue*

If not, then the tool first replaces f by f^2 or f^3, and similar for g, in order to obtain the same dominant eigenvalue

The proof requires $\tau(f^n(u_i)) = \rho(g^{n+1}(0))$ for all n which is only possible if f and g have the same *dominant eigenvalue*

If not, then the tool first replaces f by f^2 or f^3, and similar for g, in order to obtain the same dominant eigenvalue

Then the conditions of the theorem are checked, and if they hold, then the general proof of the theorem is instantiated to the specific case, yielding a proof that is readable without being aware of the theorem, as in our example

The proof requires $\tau(f^n(u_i)) = \rho(g^{n+1}(0))$ for all n which is only possible if f and g have the same *dominant eigenvalue*

If not, then the tool first replaces f by f^2 or f^3, and similar for g, in order to obtain the same dominant eigenvalue

Then the conditions of the theorem are checked, and if they hold, then the general proof of the theorem is instantiated to the specific case, yielding a proof that is readable without being aware of the theorem, as in our example

As the proof is generated by a computer program, it also may work for much larger cases where checking the conditions is very laborious, and indeed it does

The proof requires $\tau(f^n(u_i)) = \rho(g^{n+1}(0))$ for all n which is only possible if f and g have the same *dominant eigenvalue*

If not, then the tool first replaces f by f^2 or f^3, and similar for g, in order to obtain the same dominant eigenvalue

Then the conditions of the theorem are checked, and if they hold, then the general proof of the theorem is instantiated to the specific case, yielding a proof that is readable without being aware of the theorem, as in our example

As the proof is generated by a computer program, it also may work for much larger cases where checking the conditions is very laborious, and indeed it does

The origin of this research was in trying to find the smallest representation of even(fib) as a morphic sequence

Brute force computer search for a morphism g over $\{0, 1, 2, 3, 4\}$ such that $|g(a|) \leq 2$ for all a and and the first N elements of even(fib) and $\rho(g^\infty(0))$ coincide for some big number N gave

$$g(0) = 01, g(1) = 2, g(2) = 31, g(3) = 04, g(4) = 0$$

$$\rho(0) = \rho(1) = 0, \rho(2) = \rho(3) = \rho(4) = 1$$

Brute force computer search for a morphism g over $\{0,1,2,3,4\}$ such that $|g(a|) \leq 2$ for all a and and the first N elements of even(fib) and $\rho(g^\infty(0))$ coincide for some big number N gave

$$g(0) = 01, g(1) = 2, g(2) = 31, g(3) = 04, g(4) = 0$$

$$\rho(0) = \rho(1) = 0, \rho(2) = \rho(3) = \rho(4) = 1$$

for which it was easily checked that even(fib) and $\rho(g^\infty(0))$ coincide for the first million elements, so making it very likely that even(fib) $= \rho(g^\infty(0))$

Brute force computer search for a morphism g over $\{0, 1, 2, 3, 4\}$ such that $|g(a|) \leq 2$ for all a and and the first N elements of even(fib) and $\rho(g^\infty(0))$ coincide for some big number N gave

$$g(0) = 01, g(1) = 2, g(2) = 31, g(3) = 04, g(4) = 0$$
$$\rho(0) = \rho(1) = 0, \rho(2) = \rho(3) = \rho(4) = 1$$

for which it was easily checked that even(fib) and $\rho(g^\infty(0))$ coincide for the first million elements, so making it very likely that even(fib) $= \rho(g^\infty(0))$

But how to prove this?

Brute force computer search for a morphism g over $\{0,1,2,3,4\}$ such that $|g(a|) \leq 2$ for all a and and the first N elements of even(fib) and $\rho(g^\infty(0))$ coincide for some big number N gave

$$g(0) = 01, g(1) = 2, g(2) = 31, g(3) = 04, g(4) = 0$$
$$\rho(0) = \rho(1) = 0, \rho(2) = \rho(3) = \rho(4) = 1$$

for which it was easily checked that even(fib) and $\rho(g^\infty(0))$ coincide for the first million elements, so making it very likely that even(fib) $= \rho(g^\infty(0))$

But how to prove this?

Try to prove $\tau(f^\infty(0)) = \rho(g^\infty(0))$ for some more complicated f, τ for which even(fib) $= \tau(f^\infty(0))$ by construction

In trying to prove $\tau(f^\infty(0)) = \rho(g^\infty(0))$, a proof was found after g was replaced by g^3

In trying to prove $\tau(f^\infty(0)) = \rho(g^\infty(0))$, a proof was found after g was replaced by g^3

Later the proof was generalized to the theorem and the tool was developed

In trying to prove $\tau(f^\infty(0)) = \rho(g^\infty(0))$, a proof was found after g was replaced by g^3

Later the proof was generalized to the theorem and the tool was developed

It showed up to apply on many other examples

In trying to prove $\tau(f^\infty(0)) = \rho(g^\infty(0))$, a proof was found after g was replaced by g^3

Later the proof was generalized to the theorem and the tool was developed

It showed up to apply on many other examples

It does not apply on all examples

In trying to prove $\tau(f^\infty(0)) = \rho(g^\infty(0))$, a proof was found after g was replaced by g^3

Later the proof was generalized to the theorem and the tool was developed

It showed up to apply on many other examples

It does not apply on all examples

Improving the approach is a topic of ongoing research

Conclusions

Conclusions

- We gave equivalent *characterizations* of morphic sequences:

Conclusions

- We gave equivalent *characterizations* of morphic sequences:

  by automata (mix-DFAOs) and by finiteness of a particular class of subsequences

Conclusions

- We gave equivalent *characterizations* of morphic sequences:

  by automata (mix-DFAOs) and by finiteness of a particular class of subsequences

- We visualized morphic sequences by *turtle figures*

Conclusions

- We gave equivalent *characterizations* of morphic sequences:

  by automata (mix-DFAOs) and by finiteness of a particular class of subsequences

- We visualized morphic sequences by *turtle figures*

  In particular we focused on *finite* figures and *fractal* figures

Conclusions

- We gave equivalent *characterizations* of morphic sequences:

  by automata (mix-DFAOs) and by finiteness of a particular class of subsequences

- We visualized morphic sequences by *turtle figures*

  In particular we focused on *finite* figures and *fractal* figures

- We gave an approach to automatically prove that two morphic sequences are *equal*

Conclusions

- We gave equivalent *characterizations* of morphic sequences:

  by automata (mix-DFAOs) and by finiteness of a particular class of subsequences

- We visualized morphic sequences by *turtle figures*

  In particular we focused on *finite* figures and *fractal* figures

- We gave an approach to automatically prove that two morphic sequences are *equal*

- Thank you