Mahler equations for Zeckendorf numeration

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Numeration systems

A numeration system (U, B) consists of

- a sequence of natural numbers $U = (u_n)_{n\geq 0}$ with $u_0 = 1$, and
- a finite ordered digit set B such that for each $n \in \mathbb{N}$ there are $b_k, \ldots, b_0 \in B$ with $n = \sum_{i=0}^{k} b_i u_i$.

We say that $b_k \ldots b_0$ is the *canonical* representation of n if $b_k \ldots b_0$ is the greatest representation of n for the lexicographic order.
We write

$$(n)_U := b_k \ldots b_0 \cdot \text{ and } [b_k \ldots b_0]_U = n.$$

Example (The base-q numeration system, $q \in \mathbb{N}$)

$U = (q^n)_{n\geq 0}$, $B = \{0, 1, \ldots q-1\}$.
If $q = 3$, then

$$(26)_3 = 222\cdot$$

Zeckendorf numeration

Recall the Fibonacci numbers $(F_n)_{n\geq 0}$, defined by

$$F_{-2} = 0$$
$$F_{-1} = 1,$$
$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 0.$$

The *Zeckendorf* numeration system is Z$= ((F_n)_n, B = \{0,1\})$.

The canonical expansion $(n)_Z = b_n \dots b_0\cdot$ satisfies $b_i b_{i+1} = 0$ for each i.

Example

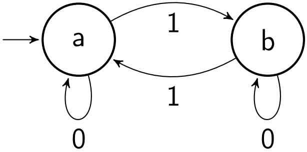
34	21	13	8	5	3	2	1	·	1	0
0	1	1	1	0	0	0	0	·		
1	0	0	1	0	0	0	0	·		

$$(42)_Z = 10010000\cdot \text{ and } [1110000]_Z = 42.$$

Automaticity base U

A sequence $(a_n)_{n\geq 0}$ taking values in a finite alphabet $\mathcal{A}$ is U-automatic if there is a deterministic finite automaton whose output is a_n when fed $(n)_U$.
If U is the base-q numeration, we will say that $(a_n)_{n\geq 0}$ is q-automatic.

Example



An automaton generating the 2-automatic Thue-Morse sequence:
If $n = 17$, then $(n)_2 = 10001$ so $a_{17} = a$
If $n = 15$, then $(n)_2 = 1111$ so $a_{15} = a$.
$a_n = a$ precisely when $(n)_2$ contains an even number of the digit 1.

Act 1 The classical base-q case

A characterisation of q-automaticity for $q = p^n$, p prime

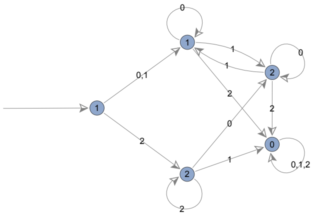
Theorem (Christol's theorem 1980)

Let $(a_n)_n$ be a sequence in $\mathbb{F}_q$, with $q = p^n$ for some n. Then $(a_n)_n$ is q-automatic iff $f(x) = \sum_{n \geq 0} a_n x^n$ is algebraic over $\mathbb{F}_q(x)$.

Example

For the Catalan numbers $(C_n)_{n\geq 0}$

- $y = \sum_{n\geq 0} C_n x^n$ satisfies $xy^2 - y + 1 = 0$ over $\mathbb{Q}$, so
- $y = \sum_{n\geq 0} C_n \bmod 3\, x^n$ satisfies $xy^2 + 2y + 1 = 0$ over $\mathbb{F}_3$, and hence $(C_n \bmod 3)_{n\geq 0}$ is automatic.



Christol's theorem, limitations

Christol's theorem does not provide a characterisation of q-automatic sequences if q not a power of a prime.

Question Is there a generalisation of Christol's theorem to all q-automatic sequences?

Answer Yes.

Question Is there a version of Christol's theorem for the Zeckendorf numeration system?

Answer Yes!

Christol's theorem, example

Let us find the annihilating polynomial for the Thue-Morse sequence, defined as

$a_n = 0$ precisely when $(n)_2$ contains an even number 1s.

$$f(x) = \sum_n a_{2n} x^{2n} + \sum_n a_{2n+1} x^{2n+1}$$
$$= \sum_n a_n x^{2n} + x \sum_n (a_n + 1) x^{2n}$$

so, with $s(x) = \frac{1}{1+x}$,

$$f(x) = (1+x) f(x^2) + x s(x^2) \;\; \& \;\; s(x) = (1+x) s(x^2) \qquad (1)$$

and

$$f(x^2) = (1+x^2) f(x^4) + x^2 s(x^4), \;\; \& \;\; s(x^2) = (1+x^2) s(x^4) \quad (2)$$

Substituting (2) in (1) we get that $f(x)$ is a root of the Ore polynomial

$$x f(x) = (1+x) f(x^2) + (1+x)^4 f(x^4).$$

q-Mahler equations

Let R be any commutative ring and let $q \geq 2$ be any natural number. Define the linear operator $\Phi : R[[x]] \to R[[x]]$ as

$$\Phi(f(x)) = f(x^q).$$

Let $A_i(x) \in R[x]$ be polynomials. The equation

$$P(x, y) = \sum_{i=0}^{d} A_i(x)\Phi^i(y) = 0$$

is called a *q-Mahler equation*.

If $f \in R[[x]]$ satisfies $P(x, f(x)) = 0$, then it is called *q-Mahler*.

If $q = p^k$, then a p^k-Mahler equation over a finite field is just a polynomial.

From automatic to regular sequences

Definition

A sequence $(a_n)_{n\geq 0}$ taking values in a finite alphabet $\mathcal{A}$ is U-automatic if there is a deterministic finite automaton whose output is a_n when fed $(n)_U$.

From automatic to regular sequences

Definition (Allouche-Shallit)

A sequence $(a_n)_{n\geq 0}$ taking values in a ~~finite alphabet $\mathcal{A}$~~ commutative ring R is ~~U-automatic~~ U-regular if there is a ~~deterministic finite automaton~~ weighted automaton whose output is a_n when fed $(n)_U$.

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Theorem (Allouche-Shallit, 1992)

A sequence is q-regular and takes on finitely many values if and only if it is q-automatic.

Examples

All from Allouche-Shallit's article, 1992:

- $a_n = \#$ 1's in $(n)_2$ defines a 2-regular sequence
- the sequence
$$0, 2, 6, 8, 20, 24, \ldots,$$
which lists the numerators of the left endpoints of the Cantor set, is 2-regular.
- $a_n = (n^j)_{n \geq 0}$ is 2-regular,
- $a_n = \sum_{i=1}^{n} \lfloor \log_a i \rfloor$ is q-regular.
- The number of comparisons required to mergesort n items,
- For $a \in \mathbb{R}$, $(a^n)_{n \geq 0}$ is q-regular if and only if $a = 0$ or a is a root of unity.

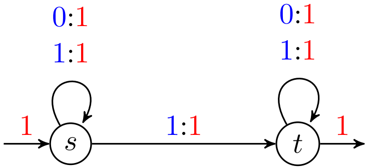
Weighted automata

A *weighted automaton* $\mathcal{A}$ with weights in the commutative ring R consists of

- a finite state set S,
- an alphabet B
- a transition weight function $\Delta : S \times B \times S \to R$ which assigns a weight to each labelled edge, denoted $s \xrightarrow{b:r} s'$, and
- initial and final weight functions $I : S \to R$ and $F : S \to R$.

Example

Let $B = \{0, 1\}$ and $R = \mathbb{F}_2$.



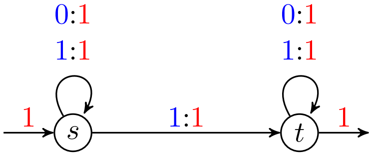
Generating sequences using weighted automata

In a weighted automaton, given a word, there may be many paths that word can follow. We are interested in the sum of the weights of all paths that this word follows.

e.g., the word 10110 follows three different paths, each of weight 1:

$$s\,t\,t\,t\,t\,t$$
$$s\,s\,s\,t\,t\,t$$
$$s\,s\,s\,s\,t\,t$$

and since $(22)_2 = 10110$ and $R = \mathbb{F}_2$, we have $u_{22} = 3 \bmod 2 = 1$.



Question: Given an automatic sequence, can one define a weighted automaton that generates it?

Theorem (Christol 1979)

Let q be a power of a prime, and let (u_n) be a sequence over $\mathbb{F}_q$. The (u_n) is q-regular if and only if it is the solution of a q-Mahler equation.

Theorem (Becker 1992, Dumas 1993)

Let $q \geq 2$, and let (u_n) be a sequence over a commutative ring R.

- *If (u_n) is q-regular sequence then it is the solution of a q-Mahler equation, and*
- *if (u_n) is the solution of an isolating q-Mahler equation, i.e., of the form $y = \sum_{i=1}^{d} A_i(x)\Phi^i(y)$, then it is q-regular.*

From isolating Mahler equations to weighted automata

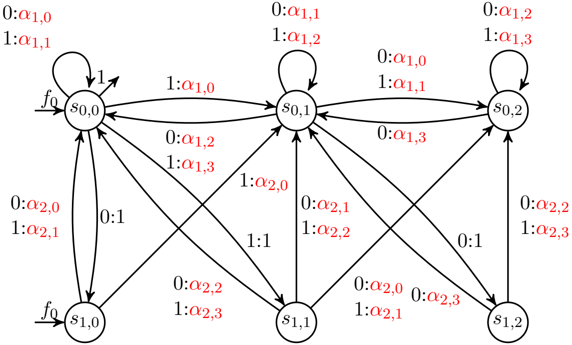
Theorem (Carton, Y, 2024)

Let $q \geq 2$ be a natural number. There exists a universal q-automaton $\mathcal{A}$, such that any isolating q-Mahler equation $P(x,y)$ over a commutative ring R with initial condition f_0 provides weights for $\mathcal{A}$, so that the corresponding weighted automaton generates the solution $f(x)$ of $P(x,y)$ with $f(0) = f_0$.

- The universal q-automaton $\mathcal{A}$ consists of a countable set of states S and a transition relation in $S \times \{0, 1, \ldots, q-1\} \times S$.
- Given an isolating q-Mahler equation $P(x,y) = y - \sum_{i=1}^{d} \left(\sum_{j=0}^{h} \alpha_{i,j} x^j \right) \Phi^i(y)$, we use its coefficients $\alpha_{i,j}$ as weights, setting other edge weights to zero, so reducing $\mathcal{A}$ to a weighted automaton.
- An important property that we use to prove this theorem is the linearity of the map $m \mapsto qm$.

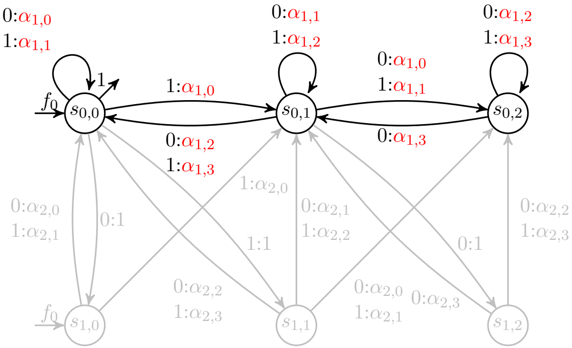
Example

A 2-Mahler equation with height 3 and exponent 2 gives



Example

whilst if the exponent drops to one we have



Act 2

Mahler equations for Zeckendorf numeration

From q-Mahler equations to Z-Mahler equations

Recall

Theorem (Becker 1992, Dumas 1993)

Let $q \in \mathbb{N}$, and let (u_n) be a sequence over a commutative ring R.

- *If (u_n) is q-regular, then it is the solution of a q-Mahler equation, and*
- *if (u_n) is a solution of an isolating q-Mahler equation, then it is q-regular.*

We prove a version of this theorem for the Zeckendorf numeration

From q-Mahler equations to Z-Mahler equations

Theorem (Carton, Y, 2024)

Let R be a commutative ring, and let (u_n) be a sequence over R.
Z-Mahler equation.

- *If (u_n) is ~~q-regular~~ Z-regular, then it is the solution of a ~~q-Mahler~~ Z-Mahler equation, and*
- *if (u_n) is a solution of an isolating ~~q-Mahler~~ Z-Mahler equation, then it is ~~q-regular~~ Z-regular.*

New Ingredients

Our proof strategy was to emulate our proof in the case of q-numeration, i.e.,

- to define the linear Z-version of the map $m \mapsto qm$, and
- to define the appropriate concept of a Z-Mahler equation,

in order to construct a weighted Z-automaton directly from an isolating Z-Mahler equation.

The Zeckendorf analogue of $n \mapsto qn$

The map $f(n) = qn$ can be written $f(n) := [w0]_q$ where $w = (n)_q$.

So, for $(n)_Z = w$, define $\phi : \mathbb{N} \to \mathbb{N}$ as

$$\phi(n) := [w0]_Z.$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\phi(n)$	0	2	3	5	7	8	10	11	13	15	16	18	20	21

Issue: ϕ is not linear.

For example,

$$3 = \phi(2) = \phi(1+1) \neq 2\phi(1) = 4.$$

Dealing with the nonlinearity of ϕ

Recall $\phi(n) := [(n)_Z\, 0]_Z$.

Define the *linearity defect δ* by

$$\delta(m,n) = \phi(m+n) - \phi(m) - \phi(n).$$

A simple application of Binet's formula gives

Lemma

For natural numbers m, n, we have $-1 \le \delta(m,n) \le 1$.

In other words, ϕ is almost linear.

We would like to track the linearity defect.

Regularity of Z-expansions, and application

Given a finite set C, consider $\mathcal{L}_C := \{w \in C^* : [w]_Z = 0\}$.

Example

Let $C = \{0, 1, -1\}$. Then the following belong to $\mathcal{L}_C$:

5	3	2	1	·	1	0
0	0	0	0	·		
1	-1	-1	0	·		
1	-1	0	-1	·	-1	

Theorem (Frougny)

For $C \subset \mathbb{Z}$ finite, there is a deterministic automaton which accepts exactly $\mathcal{L}_C$.

Corollary

There exists a deterministic automaton, which on input of $(m)_Z$ and $(n)_Z$, outputs the linearity defect $\delta(m - n, n)$ for $m \geq n \geq 0$.

Going back to our strategy

We have defined the ~~linear~~ regular map $m \mapsto \phi(m)$.

We now define the Z-version of $\Phi_q(\sum_n f_n x^n) = \sum_n f_n x^{qn}$.

Define the Z-Mahler operator $\Phi : R[[x]] \to R[[x]]$ as

$$\Phi\Big(\sum_{n \geq 0} f_n x^n\Big) := \sum_{n \geq 0} f_n x^{\phi(n)}.$$

The equation

$$P(x, y) = \sum_{i=0}^{d} A_i(x) \Phi^i(y) = 0$$

with $A_i(x) \in R[x]$, is a *Z-Mahler equation*.

If $f \in R[[x]]$ satisfies $\sum_{i=0}^{d} A_i(x) \Phi^i(f) = 0$, then it is *Z-Mahler*.

We can now prove the following by combining the classical (base-q) construction with the automaton tracking the linearity defect.

Theorem (Carton, Y, 2024)

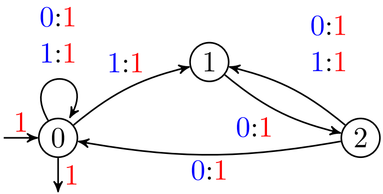
Let R be a commutative ring, and let (u_n) be a sequence over R. Z-Mahler equation. If (u_n) is Z-regular, then it is the solution of a Z-Mahler equation, and conversely, if (u_n) is a solution of an isolating Z-Mahler equation, then it is Z-regular.

Example Let

$a_n = \#$ representations of n as a sum of distinct Fibonacci numbers.

Then

$$f(x) = \sum_n a_n x^n = \prod_n (1 + x^{F_n}) \text{ and } f(x) = (1+x)\Phi(f(x)).$$



Questions

- Allouche and Shallit show that for $a \in \mathbb{R}$, $(a^n)_{n\geq 0}$ is q-regular if and only if $a = 0$ or a is a root of unity. Is there a similar result for Z-regular sequences?
- Using this, Bell, Chyzak, Coons, & Dumas characterise q-regular series in terms of the q-Mahler equations they satisfy. Is there a similar characterisation for Z-numeration?
- Adamczewski-Bell and Shäfke-Singer show that a sequence which is both k- and l-Mahler over a field of characteristic zero, with k and l multiplicatively independent, must be rational. Which series are both k- and Z-Mahler?