The level of distribution of the sum-of-digits function in arithmetic progressions

Nathan TOUMI
Université de Lorraine, IECL

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Introduction

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Every $n \in \mathbb{N}$ can be uniquely written as

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$$s_q(n) = s_q\left(\sum_{k=0}^{\infty} n_k q^k\right) := \sum_{k=0}^{\infty} n_k.$$

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Theorem A (Gelfond, 1967/1968)

Let $q, b, m \geq 2$ be integers such that $(q-1, b) = 1$. Then, for all $a \in \{0, \ldots, b-1\}$ and for all $r \in \{0, \ldots, m-1\}$, we have

$$|\{n < N \,:\, s_q(n) \equiv a \pmod{b},\ n \equiv r \pmod{m}\}| = \frac{N}{bm} + O(N^\lambda),$$

where

$$\lambda = \frac{1}{2\log(q)} \log \frac{q \sin(\pi/2b)}{\sin(\pi/2bq)} < 1.$$

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We have to estimate

$$\sum_{1 \le m \le x^\theta} \max_{\substack{y,z \ge 0 \\ z-y \le x}} \max_{0 \le r < m} \left| A_q(y,z;m,r) - \frac{z-y}{bm} \right|$$

where

$$A_q(y,z;a,b,m,r) = \left| \{ y \le n < z \,:\, s_q(n) \equiv a \pmod b, \;\; n \equiv r \pmod m \} \right|,$$

with $0 < \theta = \theta_q < 1$ as large as possible.

The level of distribution of the sum-of-digits function in base 2

We set

$$A_2(z;a,b,m,r) = \Big|\{y \le n < z \,:\, s_2(n) \equiv a \pmod b,\;\; n \equiv r \pmod m\}\Big|.$$

Theorem B (Fouvry/Mauduit, 1996)

Let $A \in \mathbb{R}$ and $D = x^{0.55771}$.
There exists $C > 0$ such that

$$\sum_{1 \le m \le D} \max_{1 \le z \le x} \max_{0 \le r < m} \left| A_2(z;a,b,m,r) - \frac{z}{bm} \right| \le Cx(\log 2x)^{-A}, \;\; x \to +\infty.$$

The level of distribution of the Thue–Morse sequence

The Thue–Morse sequence t is defined by

$$t = (t(n))_{n\in\mathbb{N}} := ((-1)^{s_2(n)})_{n\in\mathbb{N}}.$$

We set

$$A(y,z;r,m) = \Big|\{y \le n < z \,:\, t(n) = 1,\;\; n \equiv r \pmod{m}\}\Big|.$$

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Theorem C (Spiegelhofer, 2020)

Let $\varepsilon > 0$. There exists $\eta > 0$ such that

$$\sum_{1\le m\le x^{1-\varepsilon}} \max_{\substack{y,z\ge 0\\ z-y\le x}} \max_{0\le r<m} \left|A(y;z,r,m) - \frac{z-y}{2m}\right| \ll x^{1-\eta}, \qquad x \to \infty.$$

Main results

We recall that

$$A_q(y,z;a,b,m,r) = \Big|\{y \le n < z \,:\, s_q(n) \equiv a \pmod{b}, \;\; n \equiv r \pmod{m}\}\Big|.$$

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Theorem 1 (T., 2024)

Let $0 < \varepsilon < 1$. There exists a constant $C > 0$ such that

$$\sum_{1 \le m \le x^{1-\varepsilon}} \max_{\substack{y,z \\ 0 \le y < z \\ z-y \le x}} \max_{r \ge 0} \left| A_q(y,z;a,b,m,r) - \frac{z-y}{bm} \right| \le C x^{1-\eta}.$$

An admissible value for η is given for $0 < \varepsilon < 1/2$ by

$$\eta(\varepsilon) = \frac{(1-\varepsilon)\varepsilon^3 \min\left(1/4, 3\log_q\left(P^-(q)\right)\right)}{3600 \times 8^{1/\varepsilon}\left(\log(4q/\varepsilon) + 5b\log(q)/\varepsilon\right)} \times \exp\left(-\frac{5}{\varepsilon}\left(\log(4q/\varepsilon) + \frac{5b\log(q)}{\varepsilon}\right)\right).$$

The main challenge

Spiegelhofer's idea: reduce the main problem to a problem involving Gowers norms of the Thue–Morse sequence defined by

Definition: Gowers norm of the Thue–Morse sequence

The Gowers norm of the Thue–Morse is the quantity defined for $k, \rho \in \mathbb{N}$ by

$$A(k,\rho) = \frac{1}{2^{(k+1)\rho}} \sum_{\substack{0 \le n < 2^\rho \\ \mathbf{h}=(h_0,\ldots,h_{k-1}) \in \{0,\ldots,2^\rho-1\}^k}} \prod_{w=0}^{2^k-1} t(n + \mathbf{w} \cdot \mathbf{h}).$$

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Theorem D, Konieczny (2019)

There exists $\eta = \eta(k) > 0$ such that for $\rho \to +\infty$, we have

$$A(k,\rho) \ll 2^{-\eta\rho}.$$

The main challenge

Theorem 2 (T., 2024)

Let $k \geq 3$ and $\ell \in \{1, \ldots, b-1\}$.
We set

$$K = \left\lfloor \frac{\log(k)}{\log(q)} \right\rfloor + 1.$$

We define

$$\eta_0 := \frac{1}{\log(q)(K + (k+1)b)q^{(k+1)(K+b(k+1))}}.$$

Then, as $\rho \to +\infty$,

$$\frac{1}{q^{(k+1)\rho}} \sum_{\substack{0 \leq n < q^\rho \\ 0 \leq h_0, \ldots, h_{k-1} < q^\rho}} \mathrm{e}\left(\frac{\ell}{b} \sum_{\mathbf{w} = (w_0, \ldots, w_{k-1}) \in \{0,1\}^k} (-1)^{s_2(w)} s_q(n + \mathbf{w} \cdot \mathbf{h}) \right) \ll q^{-\eta_0 \rho}.$$