Non-decomposable quadratic forms over totally real fields

Magdaléna Tinková

Czech Technical University in Prague

Joint work with Pavlo Yatsyna.

June 4, 2024

Indecomposable algebraic integers

- K totally real number field
- $\mathcal{O}_K$ is the ring of algebraic integers in K
- $\mathcal{O}_K^+$ set of totally positive elements $\alpha \in \mathcal{O}_K$, i.e., all conjugates of α are positive

Indecomposable algebraic integers

- K totally real number field
- $\mathcal{O}_K$ is the ring of algebraic integers in K
- $\mathcal{O}_K^+$ set of totally positive elements $\alpha \in \mathcal{O}_K$, i.e., all conjugates of α are positive

Definition

We say that $\alpha \in \mathcal{O}_K^+$ is indecomposable in $\mathcal{O}_K$ if it cannot be written as $\alpha = \beta + \gamma$ for any $\beta, \gamma \in \mathcal{O}_K^+$.

Non-decomposable quadratic forms

- quadratic form $Q(x_1,\ldots,x_n)=\sum_{i,j=1}^{n} a_{ij}x_ix_j$ where $a_{ij}\in\mathcal{O}_K$
- Q is totally positive semi-definite if $Q(\gamma_1,\ldots,\gamma_n)\in\mathcal{O}_K^+\cup\{0\}$ for all $\gamma_i\in\mathcal{O}_K$ (TPSD)

Non-decomposable quadratic forms

- quadratic form $Q(x_1, \dots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ where $a_{ij} \in \mathcal{O}_K$
- Q is totally positive semi-definite if $Q(\gamma_1, \dots, \gamma_n) \in \mathcal{O}_K^{+} \cup \{0\}$ for all $\gamma_i \in \mathcal{O}_K$ (TPSD)

Definition

We say that TPSD Q is non-decomposable if it cannot be written as $Q = Q_1 + Q_2$ for any Q_1, Q_2 TPSD, $Q_1, Q_2 \neq 0$.

Why non-decomposable?

$K = \mathbb{Q}$

- studied by Mordell (1930, 1937) and Erdös and Ko (1938)
- the case of totally positive definite studied by Oppenheim (1946)

General fields

Theorem (Baeza and Icaza, 1995)

Let Q be a totally positive definite n-ary quadratic form over a field K. Then, there exists a constant C_{BI}, if $N(\det(Q)) \geq C_{BI}$, then Q is decomposable. Specifically, there exist a linear form L in n variables and a totally positive definite quadratic form H such that $Q = L^2 + H$.

General fields

Theorem (Baeza and Icaza, 1995)

Let Q be a totally positive definite n-ary quadratic form over a field K. Then, there exists a constant C_{BI}, if $N(\det(Q)) \geq C_{BI}$, then Q is decomposable. Specifically, there exist a linear form L in n variables and a totally positive definite quadratic form H such that $Q = L^2 + H$.

Theorem

Let Q be a totally positive definite n-ary quadratic form over a field K. Then, there exists a constant C_{TY}, if $N(\det(Q)) \geq C_{TY}$, then Q is decomposable. Specifically, there exist $\alpha \in \mathcal{O}_K^+$, a linear form L in n variables, and a totally positive semi-definite quadratic form H such that $Q = \alpha L^2 + H$.

Applications

- criterions for (non-)decomposability of quadratic forms
- estimates on the number of non-decomposable binary quadratic forms for real quadratic fields
- determination of all non-decomposable binary quadratic forms for $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6})$ and $\mathbb{Q}(\sqrt{21})$ (up to equivalence)
- some results on r-universal quadratic forms

Thank you for your attention.