# Substitutive number systems

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# Program for the next 25 minutes

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- The (classical) Dumont-Thomas numeration
- A generalisation of the Dumont-Thomas numeration
- Relations with (generalised) beta-expansions

# Letters and words

We let denote

- $\mathcal{A} := \{1, 2, \dots, \mathrm{m}\}$ a finite set (alphabet);
- $\mathcal{A}^*$ the finite words over $\mathcal{A}$;
- $\varepsilon \in \mathcal{A}^*$ the empty word;

For a word $X = \mathrm{x}_1, \dots, \mathrm{x}_n \in \mathcal{A}^*$ and a letter $\mathrm{y} \in \mathcal{A}$ we define

$$|X|_{\mathrm{y}} := \#\{j \in \{1, \dots, n\} | x_j = \mathrm{y}\}$$
$$|X| := \sum_{\mathrm{y} \in \mathcal{A}} |X|_{\mathrm{y}},$$
$$\mathbf{l}(X) := (|X|_1, |X|_2, \dots, |X|_{\mathrm{m}})^T \in \mathbb{Z}^m.$$

# Substitutions

- Let $\sigma : \mathcal{A}^* \longmapsto \mathcal{A}^*$ be a non-erasing morphism (substitution).
- Let $M_\sigma := (\mathbf{l}(\sigma(1)), \mathbf{l}(\sigma(2)), \dots, \mathbf{l}(\sigma(\mathrm{m}))) \in \mathbb{R}^{m \times m}$ be the incidence matrix. We have $\mathbf{l}(\sigma(W)) = M_\zeta \cdot \mathbf{l}(X)$ for all $X \in \mathcal{A}^*$
- We require $\sigma$ to be primitive: there exists a positive integer $n$ such that $M_\zeta^n$ is strictly positive.
- We denote by $\theta$ the (real) Perron-Frobenius eigenvalue of $M_\sigma$ (ie., $\theta > 1$) and by $\mathbf{v} \in \mathbb{Q}(\theta)^m$ a strictly positive left eigenvector with respect to $\theta$.
- We define

$$\lambda(X) : \mathcal{A}^* \longrightarrow \mathbb{R}, X \longmapsto \langle \mathbf{l}(X), \mathbf{v} \rangle.$$

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  For each $k \in \{0, \dots, n-1\}$ we have an edge $\mathtt{x} \xrightarrow{(P_k, \mathtt{x}_{k+1})} \mathtt{x}_{k+1}$.

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For a vertex $\mathrm{x} \in \mathcal{A}$ the outgoing edges can be ordered with respect to $\prec$:

$$(D_1, \mathrm{y}_1) \prec (D_2, \mathrm{y}_2) \Leftrightarrow |D_1| < |D_2| \quad (\Leftrightarrow D_1 \text{ is a prefix of } D_2).$$

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The maximal vertex is $(\mathtt{x}_1 \cdots \mathtt{x}_{n-1}, \mathtt{x}_n)$.

# Dumont-Thomas numeration

## Theorem (Dumont-Thomas, 1989)

Let $\mathsf{x} \in \mathcal{A}$. Then for each $\gamma \in [0; \lambda(\mathsf{x}))$ there exists a unique walk in the prefix graph $(D_j, \mathsf{x}_j)_{j \geq 1}$ that starts in x such that $(D_j, \mathsf{x}_j)$ is not the maximal edge for infinitely many indices $j \in \mathbb{N}$ that satisfies

$$\gamma = \sum_{j \geq 1} \lambda(D_j)\, \theta^{-j} \qquad (\sigma, \mathsf{x})\text{−expansion.}$$

# Inverse Letters

We let denote

- $\overline{\mathcal{A}} := \{\overline{1}, \overline{2}, \dots, \overline{\mathsf{m}}\}$ the set of "inverse letters";
- $\overline{\mathcal{A}}^*$ the finite words over $\overline{\mathcal{A}}$;

For a word $X = \mathsf{x}_1, \dots, \mathsf{x}_n \in \mathcal{A}^*$ we let

$$\overline{X} := \bar{\mathsf{x}}_n, \dots, \bar{\mathsf{x}}_1.$$

For a word $X = \mathsf{x}_1, \dots, \mathsf{x}_n \in \overline{\mathcal{A}}^*$ and a letter $\mathsf{y} \in \mathcal{A}$ we define

$$|X|_{\mathsf{y}} := -\#\{j \in \{1, \dots, n\} | \mathsf{x}_j = \bar{\mathsf{y}}\}$$
$$|X| := \sum_{\mathsf{y} \in \mathcal{A}} |X|_{\mathsf{y}},$$
$$\mathbf{l}(X) := (|X|_1, |X|_2, \dots, |X|_{\mathsf{m}})^T \in \mathbb{Z}^m.$$

# Coding prescriptions

## Coding Prescription

A *coding prescription (with respect to $\sigma$)* is a function $c$ with domain $\mathcal{A}$ that assigns to each letter a finite set of integers such that

- for all $\mathrm{x} \in \mathcal{A}$ we have $-|\sigma(\mathrm{x})| < k < |\sigma(\mathrm{x})|$ for all $k \in c(\mathrm{x})$.
- $c(\mathrm{x})$ is a complete set of representatives modulo $|\sigma(\mathrm{x})|$ for all $\mathrm{x} \in \mathcal{A}$, that is $\#c(\mathrm{x}) = |\sigma(\mathrm{x})|$ and for all $k, k' \in c(\mathrm{x})$ with $k \neq k'$ we have $k \not\equiv k' \mod |\sigma(\mathrm{x})|$;

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For a primitive substitution $\sigma$ and a coding prescription $c$ wrt, $\sigma$ we call the pair $(\sigma, c)$ a setting.

# Associated graph

With a setting $(\sigma, c)$ we associate the directed graph $G_{\sigma,c}$:

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  For each $k \in c(x)$, $k < 0$ we have an edge $\bar{\mathrm{x}} \xrightarrow{(\overline{S_k}, \mathrm{x}_k)} \mathrm{x}_k$.

# Ordering of edges

We define $\prec$ on the outgoing edges of a vertex $\mathsf{x} \in \mathcal{A} \cup \overline{A}$ of $G_{\sigma,c}$:

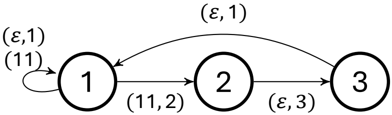
$$(D_1, \mathsf{y}_1) \prec (D_2, \mathsf{y}_2) \Leftrightarrow \begin{cases} |D_1| < |D_2| & \text{if } D_1 \neq D_2, \\ |\mathsf{y}_1| < |\mathsf{y}_2| & \text{if } D_1 = D_2. \end{cases}$$

# Example

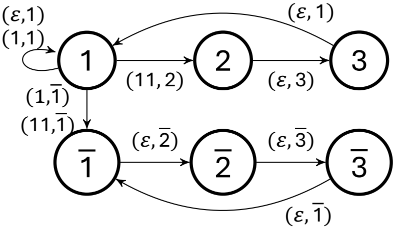
$$\sigma : 1 \mapsto 112, 2 \mapsto 3, 3 \mapsto 1$$

$$\begin{array}{llll} c_0: & 1 \mapsto \{0,1,2\}, & 2 \mapsto \{0\}, & 3 \mapsto \{0\} \\ c_1: & 1 \mapsto \{-2,-1,0\}, & 2 \mapsto \{0\}, & 3 \mapsto \{0\} \\ c_2: & 1 \mapsto \{-1,0,1\}, & 2 \mapsto \{0\}, & 3 \mapsto \{0\} \\ c_3: & 1 \mapsto \{-2,0,2\}, & 2 \mapsto \{0\}, & 3 \mapsto \{0\} \end{array}$$

Prefix graph




$G_{\sigma,c_0}$



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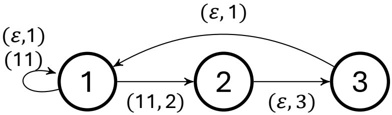
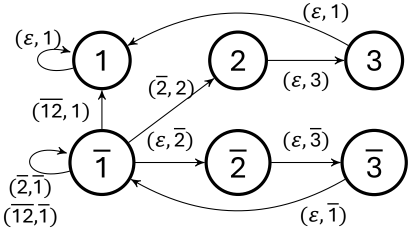



$G_{\sigma,c_1}$



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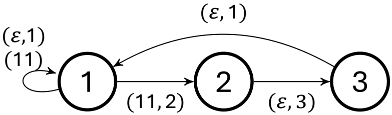
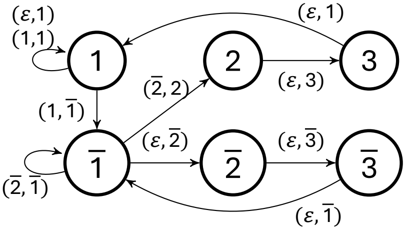



$G_{\sigma,c_2}$



# Example

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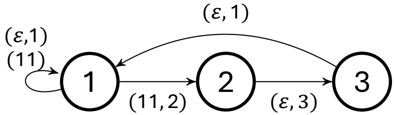
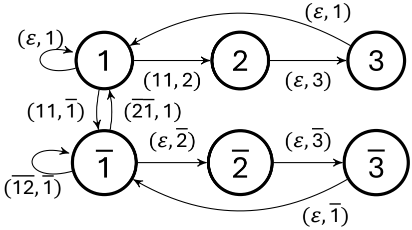



$G_{\sigma,c_3}$



# Induced sets

We are interested in the (infinite) walks on $G_{\sigma,c}$ and define for each $\mathsf{x} \in \mathcal{A} \cup \overline{A}$

$$I(\mathsf{x}) = \left\{ \sum_{j \geq 1} \lambda(D_j)\, \theta^{-j} : (D_j, \mathsf{x}_j) \text{ is a walk that starts in } \mathsf{x} \right\}.$$

The set list $\{I(\mathsf{x}) : x \in \mathcal{A} \cup \overline{A}\}$ is fixed by a graph directed iterated function system.

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The set list $\{I(\mathsf{x}) : x \in \mathcal{A} \cup \overline{A}\}$ is fixed by a graph directed iterated function system.
For all $\mathsf{x} \in \mathcal{A}$ we have

$$I(\mathsf{x}) \subset [0, \lambda(\mathsf{x})],$$
$$I(\overline{\mathsf{x}}) \subset [-\lambda(\mathsf{x}), 0].$$

The exact structure of $I(\mathsf{x})$ is determined by the coding prescription.

# Special types of settings

**Continuous setting** We say that the setting $(\sigma, c)$ is continuous if

$$\forall x \in \mathcal{A} : c(x) \text{ is a set of consecutive integers.} \tag{CS}$$

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**Even setting** We say that the setting $(\sigma, c)$ is even if

$$\forall \mathrm{x} \in \mathcal{A} : |\sigma(\mathrm{x})| \equiv 1 \mod 2 \text{ and } c(\mathrm{x}) \subset 2\mathbb{Z}. \tag{ES}$$

# Structure of $I(x)$

## Theorem

Let $\sigma$ be a primitive substitution and $c$ be a coding prescription wrt. $\sigma$. Then the following items hold for all $\mathsf{x} \in \mathcal{A}$.

- If (ES) holds then we have $I(\mathsf{x}) = [0, \lambda(\mathsf{x})]$ and $I(\overline{\mathsf{x}}) = [-\lambda(\mathsf{x}), 0]$.

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- $I(\mathsf{x}) \cup (\lambda(\mathsf{x}) + I(\overline{\mathsf{x}})) = [0, \lambda(\mathsf{x})]$ where the union is disjoint wrt. the 1-dimensional Lebesgue measure if and only if (ES) does not hold.

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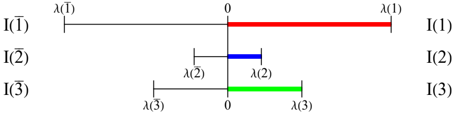
- If (CS) holds then $I(\mathsf{x})$ and $I(\bar{\mathsf{x}})$ are intervals.

# Example

$$\sigma : 1 \mapsto 1121123, 2 \mapsto 1, 3 \mapsto 112$$

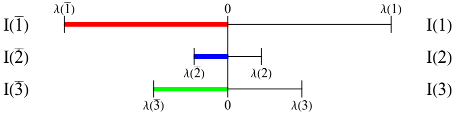
$$c_0(1) = \{0,1,2,3,4,5,6\}$$
$$c_0(2) = \{0\}$$
$$c_0(3) = \{0,1,2\}$$

(CS) satisfied




$$c(1) = \{-6,-5,-4,-3,-2,-1,0\}$$
$$c(2) = \{0\}$$
$$c(3) = \{-2,-1,0\}$$

(CS) satisfied




$$c(1) = \{-4,-3,-2,-1,0,1,2\}$$
$$c(2) = \{0\}$$
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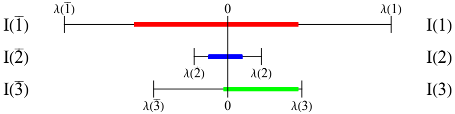

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$c(2) = \{0\}$
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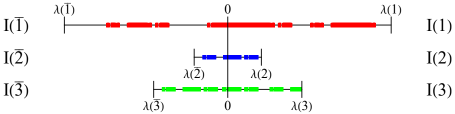


$c(1) = \{-6, -5, 0, 3, 4, 5, 6\}$
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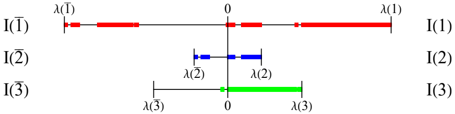


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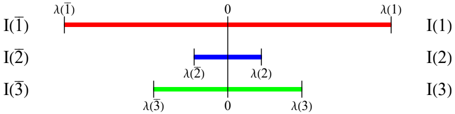

# Generalised Dumont-Thomas numeration

If $I(x) = [\alpha, \beta]$ is an interval then we let denote $\tilde{I}(x) = [\alpha, \beta)$ the corresponding right-open interval.

# Generalised Dumont-Thomas numeration

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## Theorem

Let $(\sigma, c)$ satisfy (CS) of (ES) and $\mathsf{x} \in \mathcal{A} \cup \overline{\mathcal{A}}$. Then for each $\gamma \in \tilde{I}(\mathsf{x})$ there exists a unique walk $(D_j, \mathsf{x}_j)_{j\geq 1}$ in $G(\sigma, c)$ that starts in $\mathsf{x}$ such that $(D_j, \mathsf{x}_j)$ is not the maximal edge for infinitely many indices $j \in \mathbb{N}$ that satisfies

$$\gamma = \sum_{j\geq 1} \lambda(D_j)\, \theta^{-j} \qquad (\sigma, c, \mathsf{x})\text{−expansion.}$$

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Let $c_0$ be the coding prescription (wrt. $\sigma$) that assigns to each letter a set of non-negative integers. Then for each $\mathsf{x} \in \mathcal{A}$ we have $I(\overline{\mathsf{x}}) = \{0\}$, $I(\mathsf{x}) = [0, \lambda(\mathsf{x})]$ and for each $\gamma \in \tilde{I}(\mathsf{x})$ the $(\sigma, c_0, \mathsf{x})$−expansion corresponds to the $(\sigma, \mathsf{x})$−expansion.

# Periodicity and Finiteness

Define the following properties for a setting $(\sigma, c)$ that satisfies (CS) or (ES).

$$\text{For all } \mathsf{x} \in \mathcal{A} \cup \overline{\mathcal{A}}, \gamma \in \tilde{I}(\mathsf{x}) \cap \mathbb{Q}(\theta):$$
$$\text{the } (\sigma, c, \mathsf{x})\text{−expansion is eventually periodic;} \quad \text{(P)}$$

$$\text{For all } \mathsf{x} \in \mathcal{A} \cup \overline{\mathcal{A}}, \gamma \in \tilde{I}(\mathsf{x}) \cap \mathbb{Z}[\lambda(1, \lambda(2), \dots, \lambda(\mathsf{m})]:$$
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## Theorem

Let $(\sigma, c_1)$ and $(\sigma, c_2)$ satisfy (CS) of (ES). Then

$$(\sigma, c_1) \text{ satisfies (P)} \Leftrightarrow (\sigma, c_2) \text{ satisfies (P)},$$
$$(\sigma, c_1) \text{ satisfies (F)} \Leftrightarrow (\sigma, c_2) \text{ satisfies (P)}.$$

# Generalised beta-expansion

Let $\delta \in [0,1)$, $\beta > 1$ and define the generalised beta-transformation

$$T_{\beta,\delta} : [-\delta, 1-\delta) \longrightarrow [-\delta, 1-\delta), \gamma \longmapsto \beta\gamma - \lfloor \beta\gamma + \delta \rfloor.$$

For $\gamma \in [-\delta, 1-\delta)$ let $d_{\beta,\delta}(\gamma) := (d_j)_{j\geq 1}$ with

$$d_j = \beta T_{\beta,\delta}^{j-1}(\gamma) - T_{\beta,\delta}^{j}(\gamma).$$

Then we have

$$\gamma = \sum_{j\geq 1} d_j \beta^{-j} \qquad ((\beta,\delta)\text{−expansion}).$$

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The case $\delta = 0$ corresponds to the (classical) beta-expansion (Rényi 1957).
The case $\delta = 1/2$ corresponds to the symmetric beta-expansion by (Akiyama-Scheicher 2007).

# Generalised beta-substitution

Define the left-continuous counterpart of $T_{\varepsilon,\beta}$:

$T^*_{\beta,\delta} : (-\delta, 1-\delta] \longrightarrow (-\delta, 1-\delta], \gamma \longmapsto \beta\gamma + \lfloor -\beta\gamma + 1 - \delta \rfloor$

and for $\gamma \in (-\delta, 1-\delta]$ let $d^*_{\beta,\delta}(\gamma) := (d^*_j)_{j\geq 1}$ with

$$d^*_j = \beta\, {T^*_{\beta,\delta}}^{j-1}(\gamma) - {T^*_{\beta,\delta}}^{j}(\gamma).$$

We suppose that

- $d_{\beta,\delta}(-\delta)$ is eventually periodic and consist of non-positive integers only;
- $d^*_{\beta,\delta}(1-\delta)$ is eventually periodic and consist of non-negative integers only.

# Generalised beta-substitution

W.l.o.g we may assume that $d_{\delta,\beta}(-\delta)$ and $d^*_{\delta,\beta}(1-\delta)$ have the same pre-period and the same period:

$$\begin{aligned} d_{\beta,\delta}(-\delta) = &\; -\ell_1,\dots,-\ell_q,\left(-\ell_{q+1},\dots,-\ell_{q+p}\right)^\omega, \\ d^*_{\beta,\delta}(1-\delta) = &\; \quad r_1,\dots,\quad r_q,\left(\quad r_{q+1},\dots,-r_{q+p}\right)^\omega, \end{aligned}$$

$\ell_1,\dots,\ell_{q+p},r_1,r_{q+p} \geq 0$.

We define the substitution $\sigma_{\beta,\delta}$ over the alphabet $\mathcal{A} = \{1,\dots,\mathrm{m} = p+q\}$ and the coding prescription $c_\delta$ wrt $\sigma_{\beta,\delta}$.

$$\sigma_{\beta,\delta}(x) = \begin{cases} \underbrace{1\cdots1}_{r_x}(\mathrm{x}+1)\underbrace{1\cdots1}_{\ell_x} & \text{if } \mathrm{x} \in \{1,\dots,\mathrm{m}-1\}, \\ \underbrace{1\cdots1}_{r_x}(q+1)\underbrace{1\cdots1}_{\ell_x} & \text{if } \mathrm{x} = \mathrm{m}, \end{cases}$$

$$c_\delta(\mathrm{x}) = \{-\ell_x,\dots,r_x\}.$$

# Generalised beta-expansions

## Theorem

The substitution $\sigma_{\beta,\delta}$ is primitive and the dominant root coincides with $\beta$, i.e $\theta = \beta$. If we normalize the left eigenvector $\mathbf{v}$ such that we have $\lambda(1) = 1$ (i.e. the first entry of $\mathbf{v}$ equals 1) then

- $I(\overline{1}) = [-\delta, 0]$ and for each $\gamma \in \tilde{I}(\overline{1})$ the $\beta, \delta-$expansion coincides with the $(\sigma_{\beta,\delta}, c_\delta, \overline{1})$ expansion;
- $I(1) = [0, 1-\delta]$ and for each $\gamma \in \tilde{I}(1)$ the $\beta, \delta-$expansion coincides with the $(\sigma_{\beta,\delta}, c_\delta, 1)$ expansion.

# Example

Let $\beta$ be the dominant root of $t^3 - 2t^2 - 1$ and $\delta = 0$.

$$d_{\beta,0}(0) = (0)^\omega = (0,0,0)^\omega$$
$$d^*_{\beta,0}(1) = (2,0,0)^\omega$$

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We define

$$\sigma_{\beta,0} : 1 \mapsto 112 \qquad c_0 : 1 \mapsto \{0,1,2\}$$
$$2 \mapsto 3 \qquad 2 \mapsto \{0\}$$
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For each $\gamma \in [0,1)$ the $(\beta,0)-$expansions corresponds to the $(\sigma_{\beta,0}, c_0, 1)$ expansion (cf. Fabre 1995).

# Example

Let $\beta$ be the dominant root of $t^3 - 2t^2 - 1$ and $\delta = 1/2$.

$$d_{\beta,1/2}(-1/2) == (-1, 0, 0)^\omega$$
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For each $\gamma \in [0, 1/2)$ the $(\beta, 1/2)-$expansions corresponds to the $(\sigma_{\beta,0}, c_{1/2}, 1)$ expansion.
For each $\gamma \in [-1/2, 0)$ the $(\beta, 1/2)-$expansions corresponds to the $(\sigma_{\beta,0}, c_{1/2}, \overline{1})$ expansion.

# Example

Let $\beta = \frac{5+\sqrt{21}}{2}$ and $\delta = \frac{3}{\beta+1}$.

$$d_{\beta,\delta}(-\delta) = -2, (-2, -1)^\omega$$
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$$\sigma_{\beta,\delta}: 1 \mapsto 11211 \qquad c_{\delta}: 1 \mapsto \{-2,-1,0,1,2\}$$
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## The end

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