Fiber denseness of intermediate β-shifts

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1. Intermediate β-shifts

Let $T_{\beta,\alpha}(x) = \beta x + \alpha \pmod 1$ with one discontinuity, where $x \in [0,1]$ and

$$(\beta,\alpha) \in \Delta := \{(\beta,\alpha) \in \mathbb{R}^2 \colon \beta \in (1,2) \text{ and } 0 < \alpha < 2-\beta\}$$

Kneading invariants: Denote the critical point $c = (1-\alpha)/\beta$. The orbits of points in $[0,1]$ under $T_{\beta,\alpha}$ can be coded by elements of $\{0,\ 1\}^{\mathbb{N}}$. The kneading sequence of a point x, $\tau_{\beta,\alpha}(x)$, is defined to be $(\epsilon_1\epsilon_2\cdots)$, where

$$\epsilon_i = 0 \quad \text{if} \quad T^{i-1}_{\beta,\alpha}(x) < c, \quad \text{and} \quad \epsilon_i = 1 \quad \text{if} \quad T^{i-1}_{\beta,\alpha}(x) > c.$$

When x is a preimage of c, x has two sequences

$$\tau_{\beta,\alpha}(x+) = \lim_{y\downarrow x} \tau_{\beta,\alpha}(y), \qquad \tau_{\beta,\alpha}(x-) = \lim_{y\uparrow x} \tau_{\beta,\alpha}(y),$$

where the $y's$ run through points of $[0,1]$ which are not the preimages of c. Let $k_+ = \tau_{\beta,\alpha}(c+)$ and $k_- = \tau_{\beta,\alpha}(c-)$, then (k_+, k_-) are called the kneading invariants of $T_{\beta,\alpha}$.

Intermediate β-shift: Let $\sigma \colon \{0,1\}^{\mathbb{N}} \circlearrowleft$ be the left-shift map.

Theorem (Hubbard & Sparrow, 1990)

$$\Omega_{\beta,\alpha} = \left\{\omega \in \{0,1\}^{\mathbb{N}} \colon \sigma(k_+) \preceq \sigma^n(\omega) \preceq \sigma(k_-) \text{ for all } n \in \mathbb{N}_0\right\}$$

Subshift of finite type (SFT): A subshift Ω is said to be of finite type if there exists a finite set F of forbidden words.

Theorem (Parry, 1960): Let $\beta \in (1,2)$.

The greedy β-shift ($\alpha = 0$) is a SFT if and only if k_- is periodic; the lazy β-shift ($\alpha = 2-\beta$) is a SFT if and only if k_+ is periodic.

Theorem (Li, Sahlsten & Samuel, 2016):

Let $\beta \in (1,2)$ and $\alpha \in (0, 2-\beta)$, the intermediate β-shift $\Omega_{\beta,\alpha}$ is a SFT if and only if both k_+ and k_- are periodic.

Matching property: We say $T_{\beta,\alpha}$ has matching if there exists a finite integer n such that $T^n_{\beta,\alpha}(0+) = T^n_{\beta,\alpha}(1-)$.

Self-admissible: k_+ and k_- satisfy that, $\sigma(k_+) \preceq \sigma^n(k_+)$ and $\sigma(k_-) \succeq \sigma^n(k_-)$ for all $n \geq 0$.

2. Questions

Theorem (Parry, 1960): The set of β such that its β-shift Ω_β is a SFT is dense in $(1, +\infty)$.

Question 1: Can we extend Parry's classic result to $\Omega_{\beta,\alpha}$?

For convenience, here we give some useful notations:

$$\begin{cases} \Delta(\beta) := \{(\beta,\alpha) \in \mathbb{R}^2 : 0 < \alpha < 2-\beta\} \text{ with } \beta \in (1,2) \text{ fixed}, \\ \mathcal{F} := \{(\beta,\alpha) \in \Delta : \Omega_{\beta,\alpha} \text{ is a SFT}\}, \\ \mathcal{F}(\beta) := \Delta(\beta) \cap \mathcal{F}, \end{cases}$$

Theorem (Li, Sahlsten, Samuel & Steiner, 2019): $\mathcal{F}$ is dense in Δ.

Theorem (Bruin, Carminati & Kalle, 2017):

Let β be a multinacci number of order k, then (β,α) has the same matching at time k for all $(\beta,\alpha) \in \Delta(\beta)$.

Theorem (Quackenbush, Samuel & West, 2020):

When β is a multinacci number, $\mathcal{F}(\beta)$ is dense in $\Delta(\beta)$.

Question 2: What is the relationship between SFT and matching? Do we still have fiber denseness of $\mathcal{F}(\beta)$ for general β?

3. Results on fiber denseness

$$\begin{cases} \Delta(k_+) := \{(\beta,\alpha) \in \Delta :\ k_+ \text{ is periodic and } \tau_{\beta,\alpha}(\frac{1-\alpha}{\beta}+) = k_+\}, \\ \Delta(k_-) := \{(\beta,\alpha) \in \Delta :\ k_- \text{ is periodic and } \tau_{\beta,\alpha}(\frac{1-\alpha}{\beta}-) = k_-\}, \\ \mathcal{K} := \{(\beta,\alpha) \in \Delta : \tau_{\beta,\alpha}(\frac{1-\alpha}{\beta}+) \text{ and } \tau_{\beta,\alpha}(\frac{1-\alpha}{\beta}-) \text{ are symmetric}\}, \\ \mathcal{M} := \{(\beta,\alpha) \in \Delta : T_{\beta,\alpha} \text{ has matching}\}, \\ \mathcal{F}(k_\pm) := \Delta(k_\pm) \cap \mathcal{F},\ \mathcal{M}(\beta) := \Delta(\beta) \cap \mathcal{M}, \\ I(\beta,\alpha) := \{(\beta,\alpha') \in \Delta(\beta) : T_{\beta,\alpha'} \text{ has same matching as } T_{\beta,\alpha}\}, \\ \mathcal{F}(\beta,\alpha) := I(\beta,\alpha) \cap \mathcal{F}. \end{cases}$$

Theorem 1: Let k_+ or k_- be self-admissible, then $\mathcal{F}(k_+)$ is dense in the fiber $\Delta(k_+)$; $\mathcal{F}(k_-)$ is dense in the fiber $\Delta(k_-)$.

Theorem 2: $\mathcal{F} \subsetneq \mathcal{M}$ and $\overline{\mathcal{F}(\beta)} = \overline{\mathcal{M}(\beta)}$ for any $\beta \in (1,2]$.

Remark 3: $\mathcal{F}(\beta) = \emptyset$ if and only if $\mathcal{M}(\beta) = \emptyset$.

Theorem 4: Let $(\beta,\alpha) \in \mathcal{M}$ and $T^{m-1}_{\beta,\alpha}(0+) = T^{m-1}_{\beta,\alpha}(1-)$. Write $k_+ = (10a_3\cdots a_m\cdots)$ and $k_- = (01b_3\cdots b_m\cdots)$. Then

1. $I(\beta,\alpha)$ is a subinterval of $\Delta(\beta)$.
2. $\overline{\mathcal{F}(\beta,\alpha)} = \overline{I(\beta,\alpha)}$.
3. the endpoints of $I(\beta,\alpha)$ can be full characterized.

Corollary 5: Let $\alpha \in (0, 2-\beta)$. $I(\beta,\alpha) = \Delta(\beta)$ if and only if β is a multinacci number.

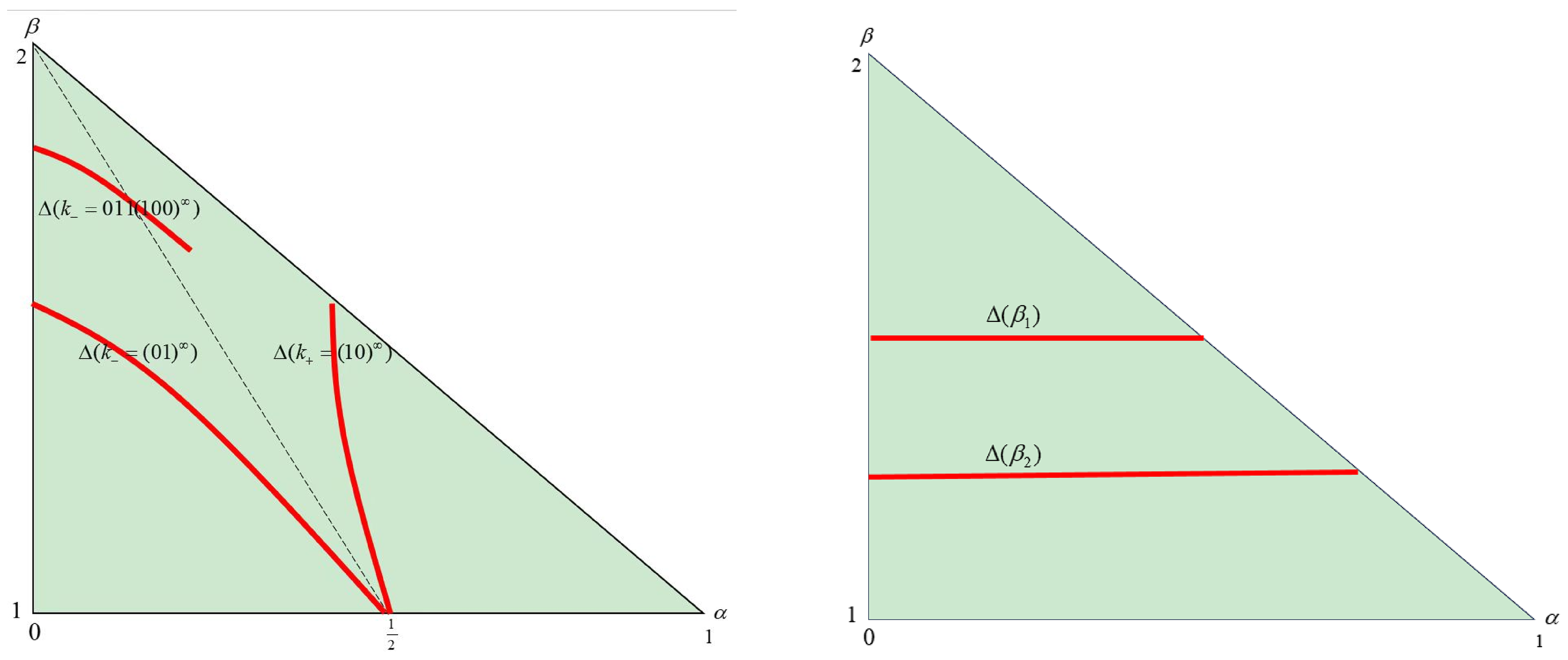

Figure 1: General cases of $\Delta(k_\pm)$, Cases of $\Delta(\beta)$.

4. Future work

(1) If β is a Pisot number, $\overline{\mathcal{F}(\beta)} = \overline{\mathcal{M}(\beta)} = \overline{\mathcal{S}(\beta)} = \overline{\Delta(\beta)}$?

(2) In which case, $\mathcal{M}(\beta) = \mathcal{F}(\beta) = \emptyset$?

Reference: Sun Y, Li B and Ding Y 2023 Fiber denseness of intermediate β-shifts of finite type *Nonlinearity* 36 5973-5997