Fiber denseness of intermediate β-shifts of finite type

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Outline

- Backgrounds

- Main resluts

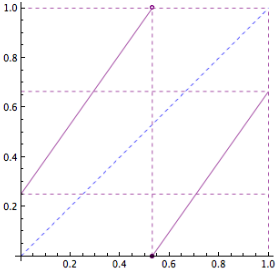
1. Backgrounds

Intermediate β-transformation

Let $T_{\beta,\alpha}(x) = \beta x + \alpha \pmod 1$ for $x \in [0,1]$. Then $T_{\beta,\alpha}$ has one discontinuity if

$$(\beta,\alpha) \in \Delta := \{(\beta,\alpha) \in \mathbb{R}^2 \colon \beta \in (1,2) \text{ and } 0 < \alpha < 2-\beta\}$$

Moreover, denote the critical point $c = (1-\alpha)/\beta$.



Kneading invariants

Kneading sequence of a point x, $\tau_{\beta,\alpha}(x)$, is defined to be $(\epsilon_1\epsilon_2\cdots)$, where

$$\epsilon_i = 0 \quad \text{if} \quad T^{i-1}_{\beta,\alpha}(x) < c, \quad \text{and} \quad \epsilon_i = 1 \quad \text{if} \quad T^{i-1}_{\beta,\alpha}(x) > c.$$

When x is a preimage of c, x has two sequences

$$\tau_{\beta,\alpha}(x+) = \lim_{y \downarrow x} \tau_{\beta,\alpha}(y), \qquad \tau_{\beta,\alpha}(x-) = \lim_{y \uparrow x} \tau_{\beta,\alpha}(y),$$

where the $y's$ run through points of $[0,1]$ which are not preimages of c.

- Kneading invariants of $\Omega_{\beta,\alpha}$ is defined to be the pair of sequences $(k_+, k_-) = (\tau_{\beta,\alpha}(c+), \tau_{\beta,\alpha}(c-))$.

Subshift of finite type

- Theorem (Hubbard & Sparrow, 1990)

$$\Omega_{\beta,\alpha} = \left\{\omega \in \{0,1\}^{\mathbb{N}} : \sigma(k_+) \preceq \sigma^n(\omega) \preceq \sigma(k_-) \text{ for all } n \in \mathbb{N}_0\right\}$$

- SFT: A subshift Ω is said to be of finite type if it can be defined by a finite set of forbidden blocks.

- Theorem (Parry, 1960) Let $\beta \in (1,2)$.
  1. The greedy β-shift ($\alpha = 0$) is a SFT if and only if k_- is periodic;
  2. The lazy β-shift ($\alpha = 2 - \beta$) is a SFT if and only if k_+ is periodic.

- Theorem (Li, Sahlsten & Samuel, 2016)
  Let $\beta \in (1,2)$ and $\alpha \in (0, 2-\beta)$, the intermediate β-shift $\Omega_{\beta,\alpha}$ is a SFT if and only if both k_+ and k_- are periodic.

Questions

Notations:

$$\begin{cases} \Delta(\beta) := \{(\beta,\alpha) \in \mathbb{R}^2 : 0 < \alpha < 2-\beta\} \text{ where } \beta \in (1,2) \text{ is fixed}, \\ \mathcal{F} := \{(\beta,\alpha) \in \Delta : \Omega_{\beta,\alpha} \text{ is a SFT}\}, \\ \mathcal{F}(\beta) := \Delta(\beta) \cap \mathcal{F}, \end{cases}$$

- Theorem (Parry, 1960)
  The set of β such that Ω_β is a SFT is dense in $(1, +\infty)$.
- Question 1:
  Can we extend Parry's classic result to $\Omega_{\beta,\alpha}$?
- Theorem (Li, Sahlsten, Samuel & Steiner, 2019)
  $\mathcal{F}$ is dense in Δ.
- Theorem (Quackenbush, Samuel & West, 2020)
  When β is a multinacci number, $\mathcal{F}(\beta)$ is dense in $\Delta(\beta)$.
- Question 2: What about general Pisot numbers?

2. Main results

Results on $\Delta(k_\pm)$

$$\begin{cases} \Delta(k_+) := \{(\beta,\alpha)\in\Delta : \ k_+ \text{ is periodic and } \tau_{\beta,\alpha}(\frac{1-\alpha}{\beta}+) = k_+\}, \\ \Delta(k_-) := \{(\beta,\alpha)\in\Delta : \ k_- \text{ is periodic and } \tau_{\beta,\alpha}(\frac{1-\alpha}{\beta}-) = k_-\}, \\ \mathcal{F}(k_\pm) := \Delta(k_\pm)\cap\mathcal{F}. \end{cases}$$

Self-admissible: k_+ and k_- satisfy that, $\sigma(k_+) \preceq \sigma^n(k_+)$ and $\sigma(k_-) \succeq \sigma^n(k_-)$ for all $n \geq 0$.

Theorem 1 (S.-Li-Ding, Nonlinearity, 2023)

Let k_+ and k_- be self-admissible. Then $\mathcal{F}(k_+)$ is dense in the fiber $\Delta(k_+)$ and $\mathcal{F}(k_-)$ is dense in the fiber $\Delta(k_-)$.

Results on $\Delta(k_\pm)$

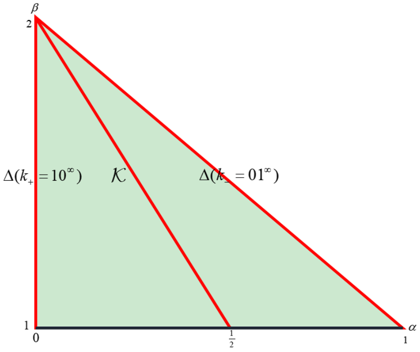
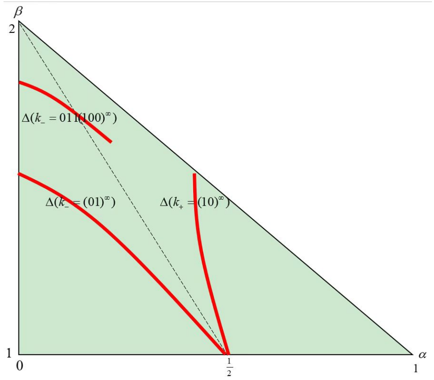


Figure: Three special cases.



General cases of $\Delta(k_\pm)$.

Matching property

- $T_{\beta,\alpha}$ or (β,α) has matching if there exists a finite integer n such that $T^n_{\beta,\alpha}(0+) = T^n_{\beta,\alpha}(1-)$, i.e., $T^{n+1}_{\beta,\alpha}(c+) = T^{n+1}_{\beta,\alpha}(c-)$.

$$\begin{cases} \mathcal{M} := \{(\beta,\alpha) \in \Delta : T_{\beta,\alpha} \text{ has matching}\}, \\ \mathcal{M}(\beta) := \Delta(\beta) \cap \mathcal{M}. \end{cases}$$

Theorem 2 (S.-Li-Ding, Nonlinearity, 2023)

$\mathcal{F} \subsetneq \mathcal{M}$, and $\overline{\mathcal{F}(\beta)} = \overline{\mathcal{M}(\beta)}$.

Results on $\Delta(\beta)$

Notations:

$$\begin{cases} I(\beta,\alpha) := \{(\beta,\alpha') \in \Delta(\beta) : T_{\beta,\alpha'} \text{ has same matching as } T_{\beta,\alpha}\}, \\ \mathcal{F}(\beta,\alpha) := I(\beta,\alpha) \cap \mathcal{F}. \end{cases}$$

Theorem 3 (S.-Li-Ding, Nonlinearity, 2023)

Let $(\beta,\alpha) \in \mathcal{M}$. Then

1. $I(\beta,\alpha)$ is a subinterval of $\Delta(\beta)$ and can be fully characterized.
2. $\overline{\mathcal{F}(\beta,\alpha)} = \overline{I(\beta,\alpha)}$.

Corollary (S.-Li-Ding, Nonlinearity, 2023)

Let $\alpha \in (0, 2-\beta)$. $I(\beta,\alpha) = \Delta(\beta)$ if and only if β is a multinacci number.