Hyperbolic Multiscale Tilings, Partitions and Numeration Systems

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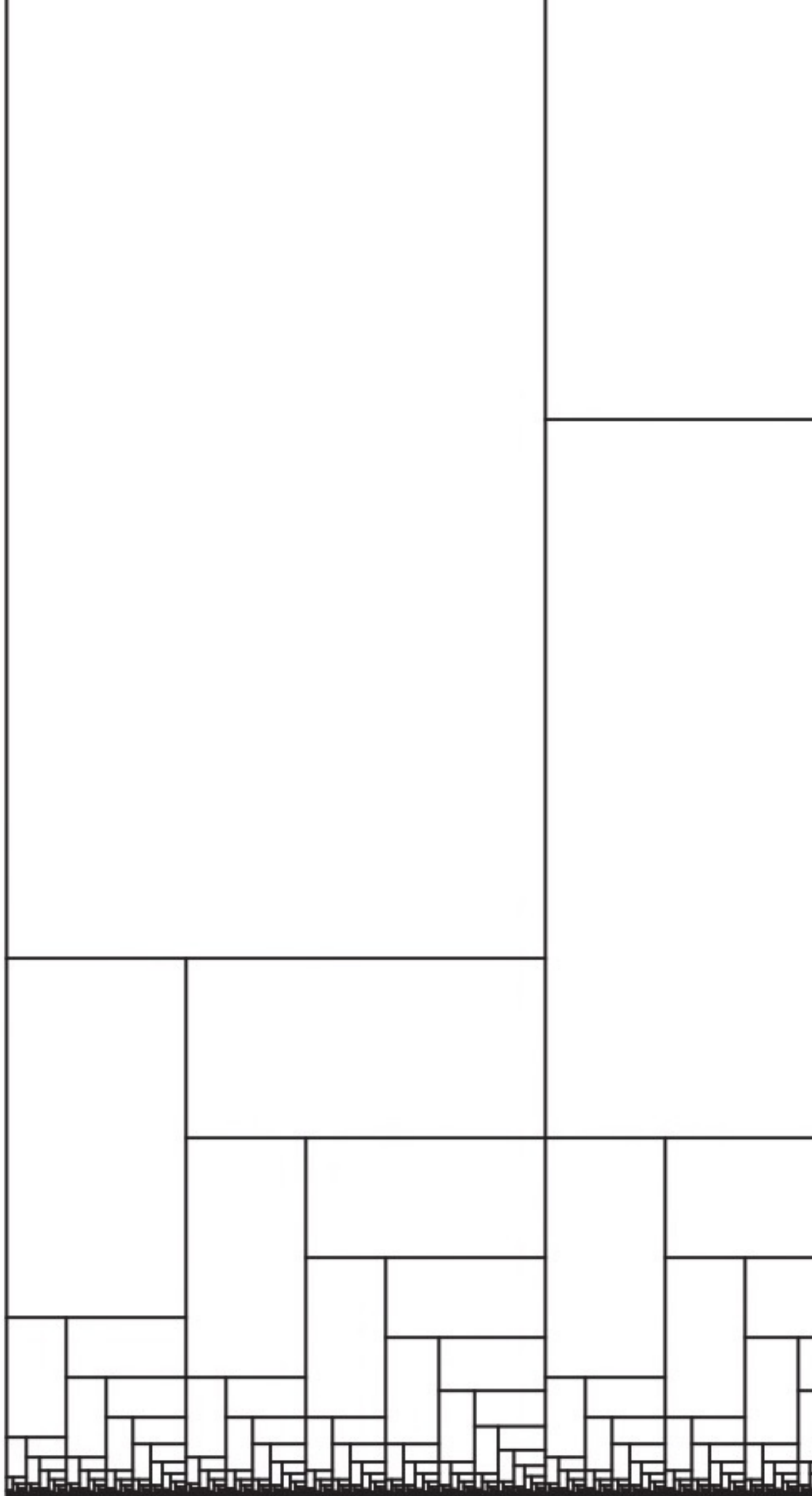
Numeration 2024, Utrecht

Based on joint work with Yaar Solomon

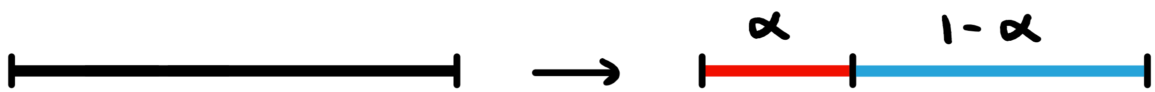
Plan of Talk

- Multiscale Substitutions
- Hyperbolic Tilings
- Statistics and Flows



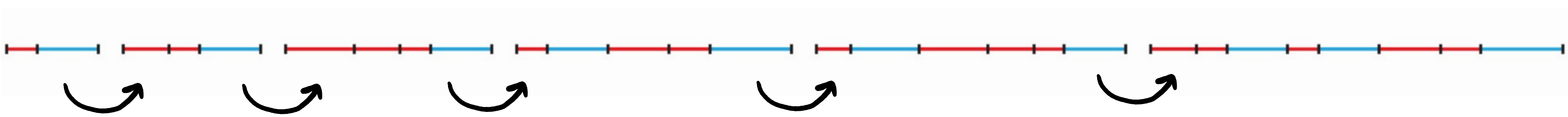
Multiscale Substitutions

Fix $0<\alpha<1$ and consider the substitution rule σ:



- multiscale substitution tilings [SS '21]

a time t dependent substitution semi-flow F_t : inflate by e^t and subsitute tiles of volume > 1



inflate with time, substitute largest tiles only

limit $\longrightarrow$

$\in X_\alpha$

$$\frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} \notin \mathbb{Q} \implies$$ aperiodic non-linearly almost repetitive tiling

incommensurability

(∞ scales nicely distributed, non-BD, minimal, uniquely erg.,...)

Multiscale Substitutions

- α-words

$\in X_\alpha$

(temporary name)

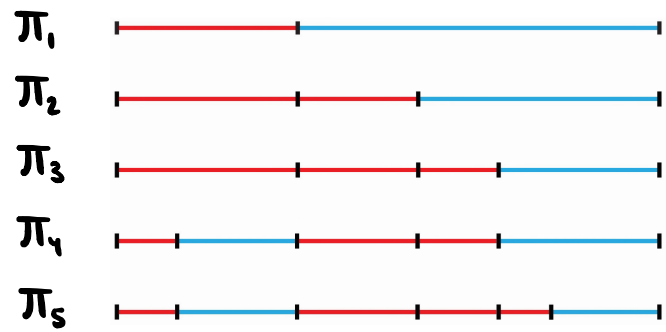
projection $\downarrow$

R B R R R B R R R B R R B R B R R B $\in \Omega_\alpha$

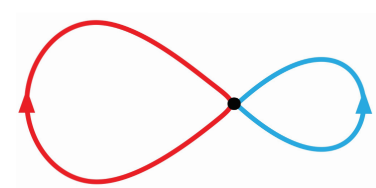
- sequences of partitions

[Kakutani '76, S '20]

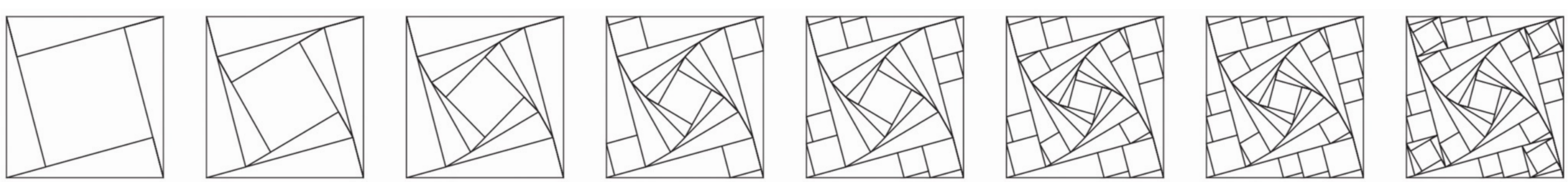
always split longest interval



- the associated graph [KSS '20]

$\log \frac{1}{\alpha}$  $\log \frac{1}{1-\alpha}$

- higher dimensions and additional prototiles



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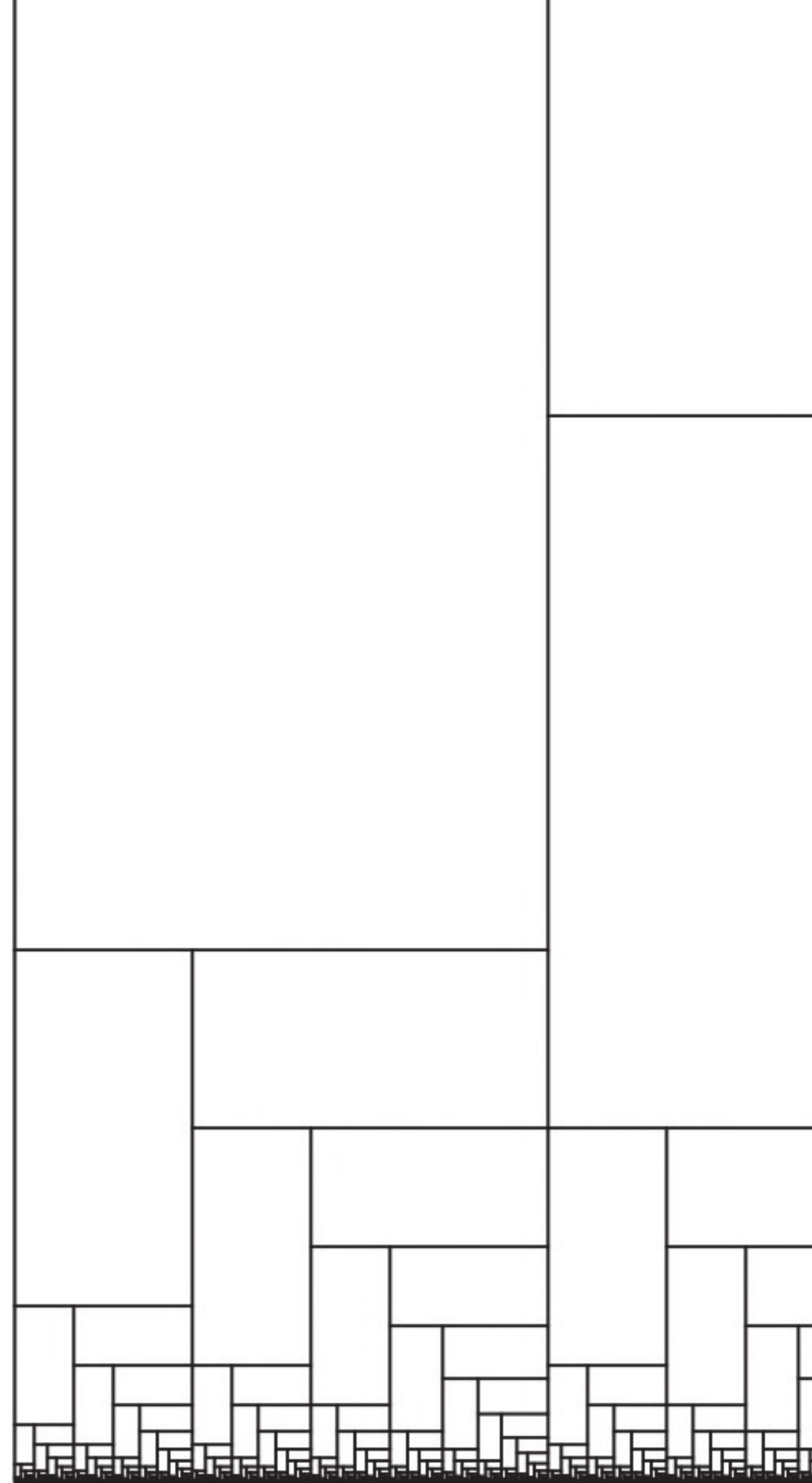


From Substitutions in $\mathbb{R}^d$ to Tilings of $\mathbb{H}^{d+1}$

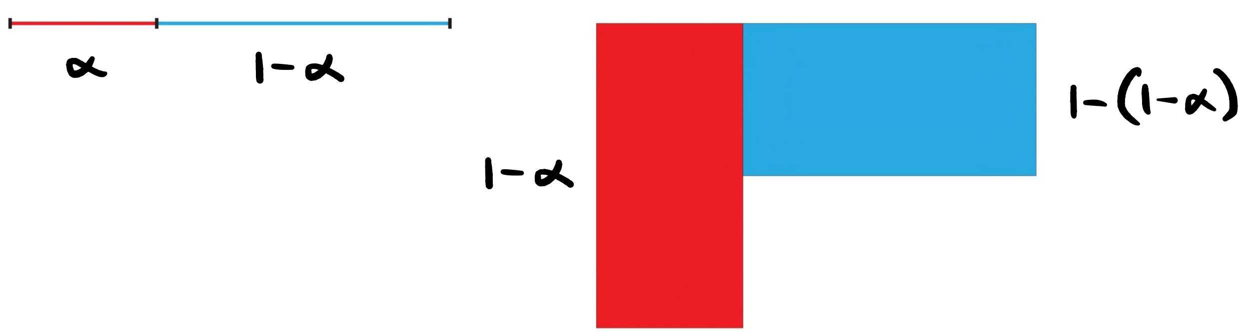
- upper half-space $\mathbb{H}^{d+1} = \{(x,s) : x = (x_1, \dots, x_d) \in \mathbb{R}^d, s > 0\}$

Two continuous actions by hyperbolic isometries:

- horospheric $\mathbb{R}^d$-action $h_y(x,s) = (y+x, s)$ for $y \in \mathbb{R}^d$
- geodesic $\mathbb{R}$-action $g_t(x,s) = (e^t x, e^t s)$ for $t \in \mathbb{R}$

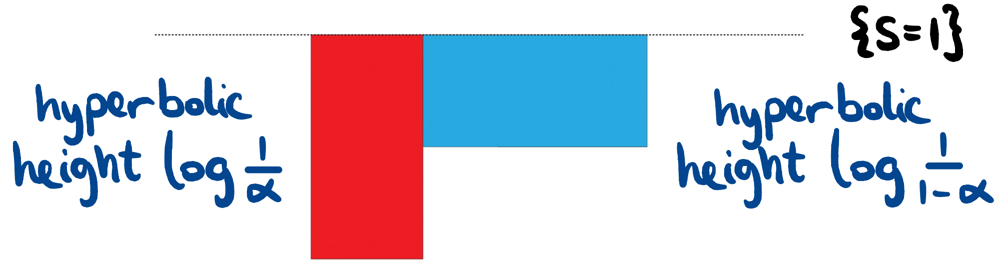
satisfying $g_t \circ h_y = h_{e^t y} \circ g_t$

- Hyperbolic tilings given a substitution σ in $\mathbb{R}^d$, give every Euclidean tile a height corresponding to its scale

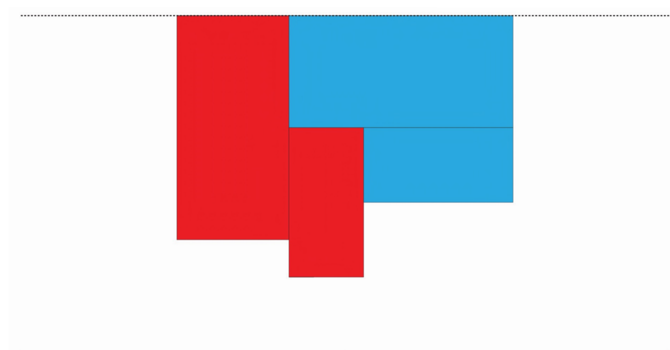


Gluing Procedure

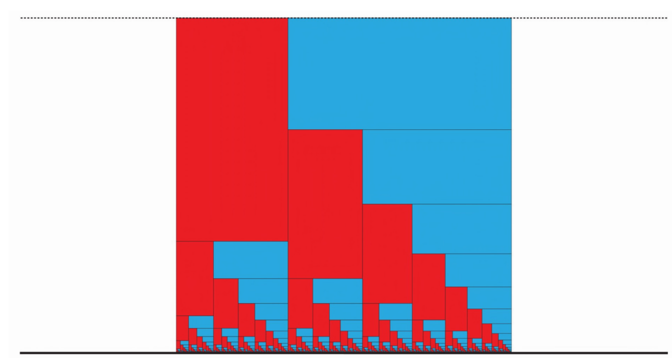
- position the patch in $\mathbb{H}^{d+1}$ aligned to the horosphere $\{s=1\}$



- glue an isometric copy of the patch to the bottom of a tile, repeat



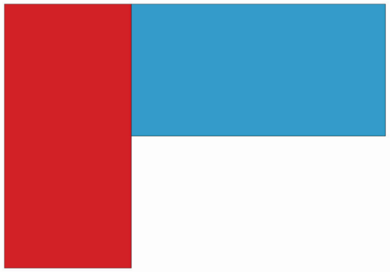
- the limit object is called a tower

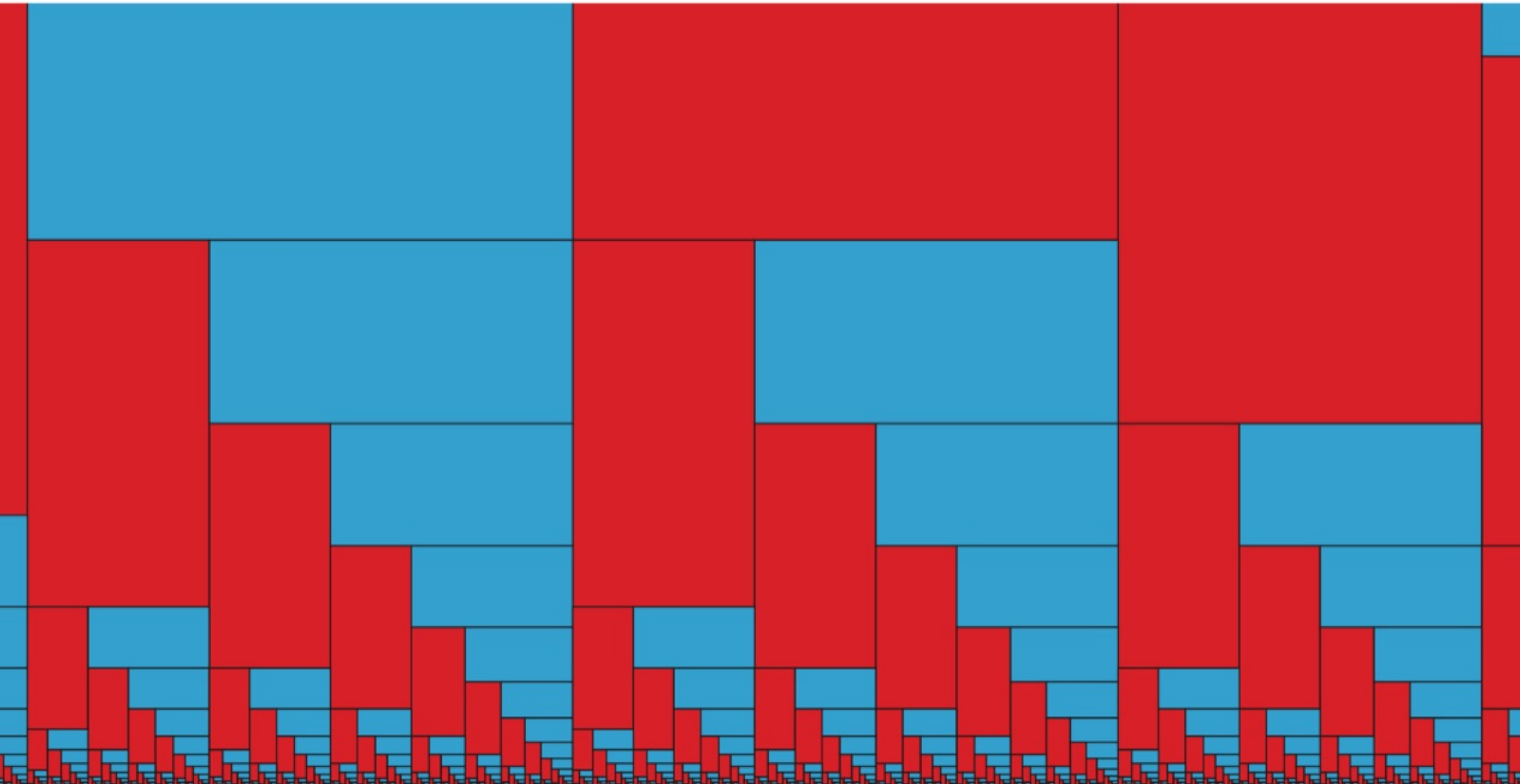


- tilings of $\mathbb{H}^{d+1}$ are partial limits of towers under g_t as $t \to \infty$.

The tiling space X_{σ}^{hyp} is the collection of all such limits.

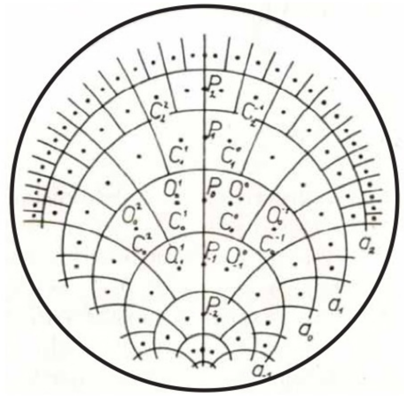
Tiling of the Hyperbolic Half-space



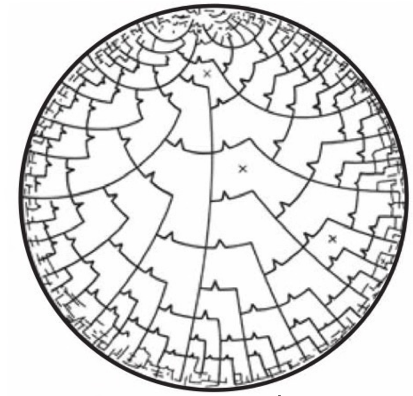


Related Hyperbolic Constructions

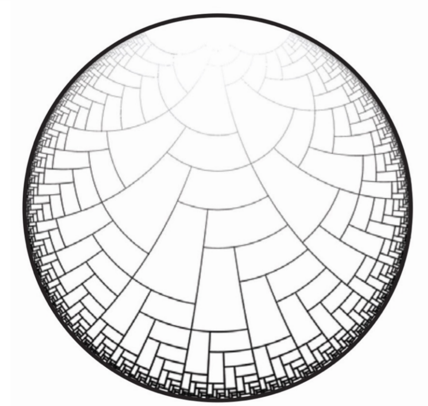
Poincaré disk model



$\alpha = \frac{1}{2}$ [Böröczky '74][1]

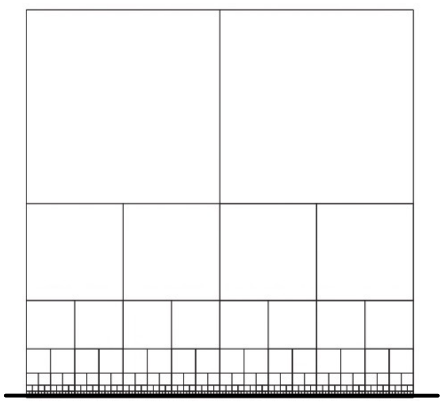


$\alpha = 1 - \frac{1}{\varphi}$ [Penrose '79][2]

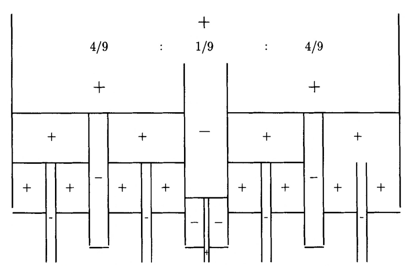


$\alpha = \frac{1}{3}$

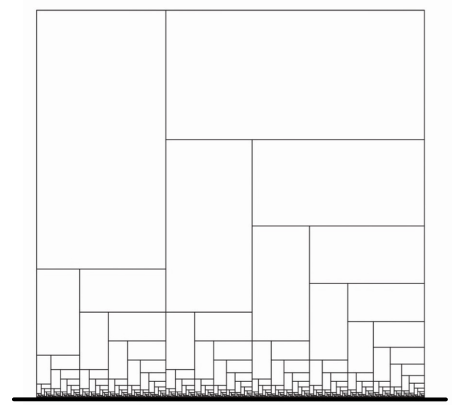
Upper half-plane model



Binary tiling



Numeration systems [Kamae '05][3]



$\alpha = \frac{1}{3}$

see also Escher's Regular Division of The Plane VI '57

1. Böröczky K., Gömbkitöltések állandó görbületű terekben I, Matematikai Lapok 25 (1974)
2. Penrose R., Pentaplexity: a class of nonperiodic tilings of the plane, Math. Intelligencer 2(1) (1979)
3. Kamae T., Numeration systems, fractals and stochastic processes, Israel J. Math. 149 (2005)

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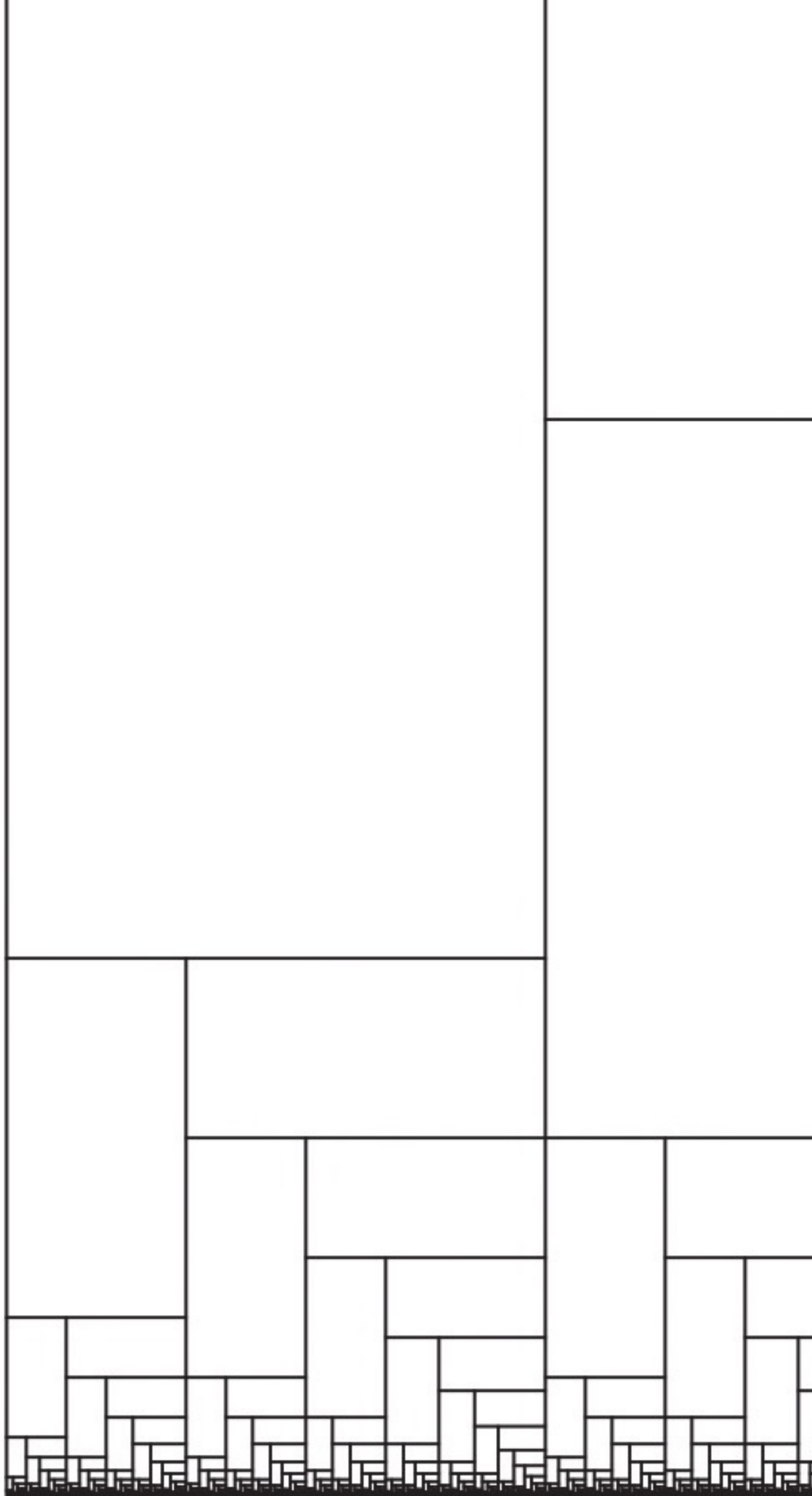
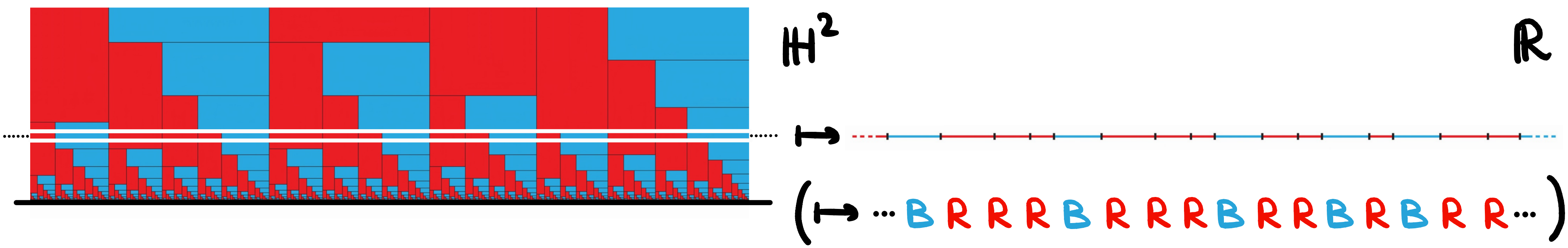


Liftings of Multiscale Substitution Tilings

Theorem Let $T \in X_\sigma^{hyp}$ be a $d+1$-dimensional hyperbolic tiling, then the map $T \mapsto T \cap \{s=1\}$ defines a d-dimensional multiscale substitution tiling generated by σ.



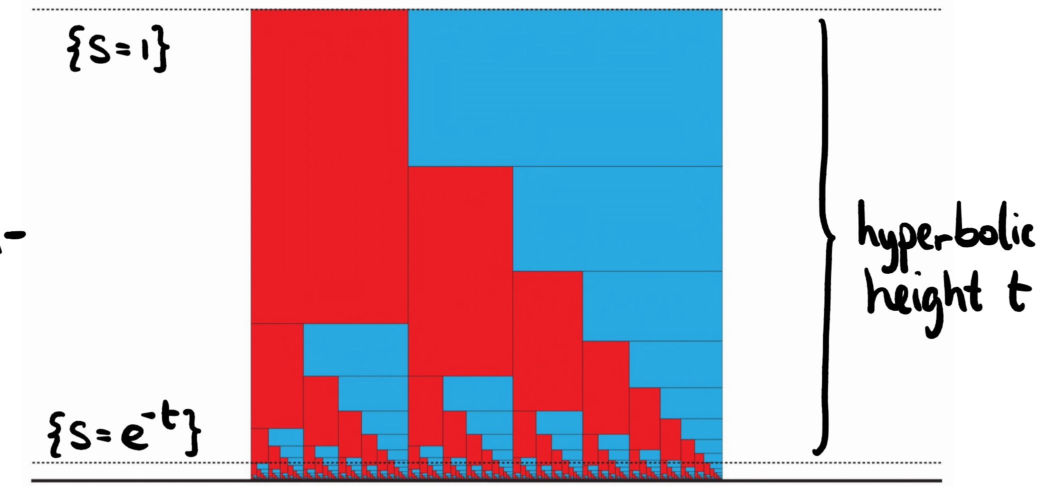
Finitely many tiles up to hyperbolic isometries

Infinitely many tiles up to Euclidean isometries (assuming incommensurability)

- horospheric $\mathbb{R}^d$-action h_y → translation action in $\mathbb{R}^d$ "space"
- geodesic $\mathbb{R}$-action g_t → substitution semiflow F_t "time"

Counting in Towers

Theorem Let σ be an incommensurable substitution in $\mathbb{R}^d$. Then



$$\#\left\{\begin{matrix}\text{tiles of type } j \text{ above} \\ \{s=e^{-t}\} \text{ inside a tower}\end{matrix}\right\} \sim \frac{[v^T \mathbb{1}]_j}{v^T H_\sigma \mathbb{1}} \cdot e^{dt}, \quad t \to \infty$$

(Paths in G_σ of length $\leq t$)

$$\#\left\{\begin{matrix}\text{tiles of type } j \text{ intersecting} \\ \{s=e^{-t}\} \text{ inside a tower}\end{matrix}\right\} \sim \frac{[v^T (S_\sigma - V_\sigma)\mathbb{1}]_j}{v^T H_\sigma \mathbb{1}} \cdot e^{dt}, \quad t \to \infty$$

(Walks in G_σ of length $= t$)

combinatorics matrix $\quad (S_\sigma)_{ij} = \sum_{T \text{ of type } j \text{ in } T_i} 1$

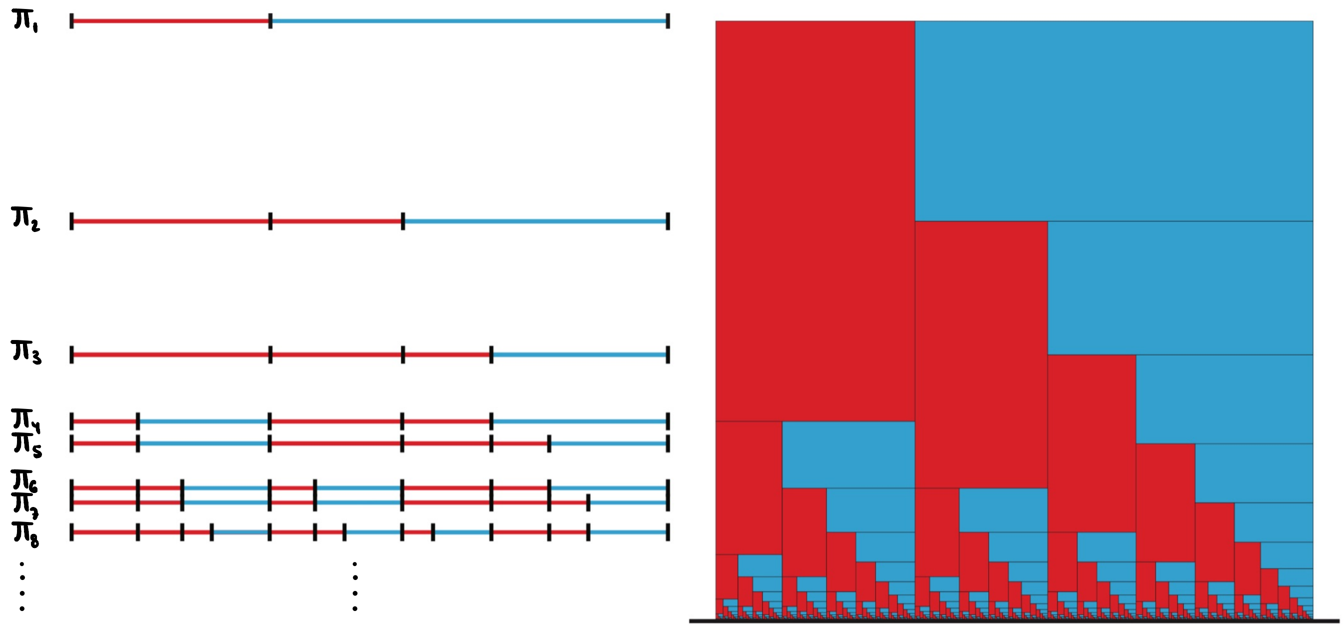
entropy matrix $\quad (H_\sigma)_{ij} = \sum_{T \text{ of type } j \text{ in } T_i} -\mathrm{vol}(T) \cdot \log \mathrm{vol}(T)$

volume matrix $\quad (V_\sigma)_{ij} = \sum_{T \text{ of type } j \text{ in } T_i} \mathrm{vol}(T)$

v^T = left Perron-Frobenius eigenvector of V_σ

Counting in Towers

Towers as suspended normalised Kakutani partitions



$$\lim_{m\to\infty} \frac{\#\,\text{red intervals in } \pi_m}{\#\,\text{intervals in } \pi_m} = \frac{2}{3} = \frac{\iint \frac{dx\,ds}{s^2}}{\iint \frac{dx\,ds}{s^2}}$$

$$\lim_{m\to\infty} \text{length}(\text{union of red intervals in } \pi_m) = \frac{-\frac{1}{3}\log\frac{1}{3}}{-\frac{1}{3}\log\frac{1}{3}-\frac{2}{3}\log\frac{2}{3}} = \frac{\iint \frac{dx\,ds}{s}}{\iint \frac{dx\,ds}{s}}$$

The Horospheric Flow

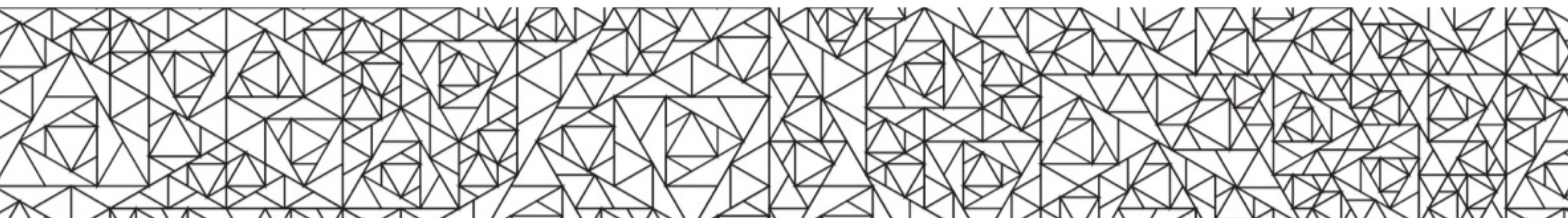
horospheric $\mathbb{R}^d$-action: $h_y(x,s) = (y+x,s)$ for $y \in \mathbb{R}^d$

Let σ be an incommensurable substitution in $\mathbb{R}^d$. Then

Theorem The dynamical system (X_σ^{hyp}, h_y) is minimal, that is, every orbit is dense.

Theorem Tilings in X_σ^{hyp} have no horospheric periods, that is, if $T \in X_\sigma^{hyp}$ and $y \in \mathbb{R}^d$ satisfy $h_y(T) = T$ then $y = 0$.

$\Rightarrow$ Aperiodic order

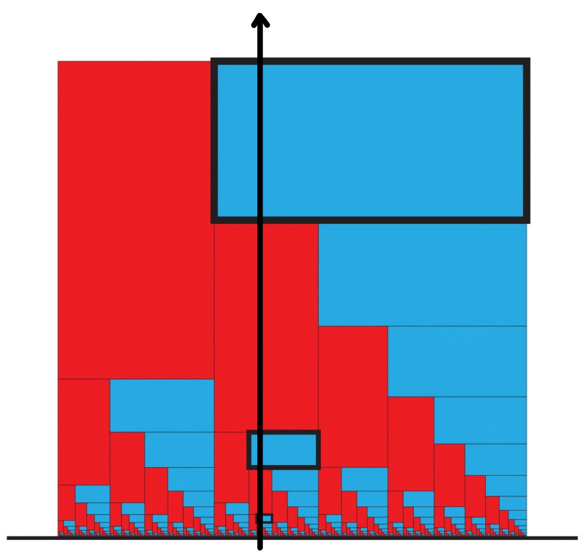


The Geodesic Flow

geodesic $\mathbb{R}$-action: $g_t(x,s) = (e^t x, e^t s)$ for $t \in \mathbb{R}$

Let σ be an incommensurable substitution in $\mathbb{R}^d$. Then

Theorem The dynamical system (X_σ^{hyp}, g_t) has dense orbits, periodic orbits (and orbits that are neither).



tiles intersecting the s-axis define words in the tile alphabet

Theorem (Prime orbit theorem following [Parry, Pollicott '83])

$$\pi_\sigma(t) = \#\{\text{periodic orbits } \tau \text{ with minimal period } \lambda(\tau) \leq t\} \sim \frac{e^{dt}}{dt}, \quad t \to \infty$$

- Tiling zeta function $\zeta_\sigma(s) := \prod_\tau (1 - e^{-\lambda(\tau)s})^{-1} = \dfrac{1}{\det(I - M_\sigma(s))} = \dfrac{1}{1 - \alpha^s - (1-\alpha)^s}$

Thank You!