Aperiodic sequences: Complexity and Rauzy fractals

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- Gohlke, Mitchell, Rust and Samuel. Rauzy fractals of random substitutions arxiv.org/abs/2401.06732 (2024)
- Gohlke, Mitchell, Rust and Samuel. Measure theoretic entropy of random substitution subshifts *Ann. Henri Poincaré* **24** 277–323 (2023)
- Gröger, Kesseböhmer, Mosbach, Samuel and Steffens. A classification of aperiodic order via spectral metrics and Jarnìk sets *Ergod. Dyn. Sys.* **39**, 3031–3065 (2019)

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Prototypical mathematical models for quasicrystals are Sturmians

Let $\theta \in (0,1) \setminus \mathbb{Q}$. For $n \in \mathbb{N}_0$ set $s_n = \lfloor n\theta \rfloor - \lfloor (n-1)\theta \rfloor$. The infinite word $s_\theta = (s_0, s_1, \ldots)$ is the Sturmian word of slope θ.

Dynamical properties of Sturmians [Hedlund and Morse, *Amer. J. Math.* **61** (1940)]

The orbit closure X_θ of s_θ under the left-shift is minimal and aperiodic with zero topological entropy.

Example

$$\theta = (-1 + \sqrt{5})/2 = [0; \overline{1}] = [0; 1, 1, 1, \ldots]$$

$$s_\theta = (1,0,1,0,1,1,0,1,0,1,1,0,1,1,0,1,0,1,1,0,1,0,1,1,0,1,1,0,1,0,1,1,0,1,\ldots)$$

$$\eta = [0; 1, 1, 1, 1, 2, 3, 6, 16, 67, 547, 4062, \ldots\ldots]$$

$$s_\eta = (1,0,1,0,1,1,0,1,0,1,1,0,\ldots)$$

Notation

$$[0; a_1, a_2, a_3 \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

Question What are measures of complexity beyond topological entropy?

Return times: The distance between occurrences of a given string (subword) of length n in s_θ.

Theorem [Duran. *Ergodic Theory Dynam. Systems* **20** (2000)]

The following are equivalent.

- The return time of occurrences of a given string of length n in s_θ is $O(n)$.
- There exists $K \in \mathbb{N}$ with $a_n \leq K$ for all $n \in \mathbb{N}$. Namely, θ is badly approximable.

Question What if we are not badly approximable?

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$$\eta = [0;1,1,1,1,2,3,6,16,\; 67,547,4062,\dots\dots]$$

$$s_\eta = (1,0,1,0,1,1,0,1,0,1,1,0,\dots)$$

Theorem [Gröger, Kesseböhmer, Mosbach, S. and Steffens. *Ergodic Theory Dynam. Systems* (2019)]

We have control over the return times, if and only if, we have control over the growth rate of the continued fraction entries.

Let $\theta = [0;a_1,a_2\dots] \in (0,1)\setminus\mathbb{Q}$ and let $\alpha \geq 1$. The following are equivalent.

- The return time between occurrences of strings of length n in s_θ is $O(n^\alpha)$.
- $0 < \limsup_{n\to\infty} a_n q_{n-1}^{1-\alpha} < \infty$

Question How large is the set of such θ?

Notation

$$[0;a_1,a_2,a_3\dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

$p_n/q_n = [0;a_1,\dots,a_n]$ with $\gcd(p_n,q_n)=1$

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Theorem [Gröger, Kesseböhmer, Mosbach, S. and Steffens. *Ergodic Theory Dynam. Systems* (2019)]

For $\alpha > 1$, letting Θ_α denote the set of such θ, we have that $\dim_{\mathcal{H}}(\Theta_\alpha) = 2/(\alpha + 1)$.

Prototypical mathematical models for quasicrystals are Sturmians

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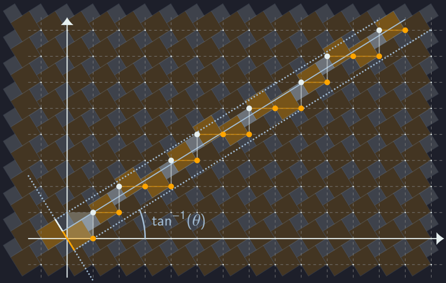
- Sturmian words are 1-balanced. For any two subwords $(\omega_0, \dots \omega_{n-1})$ and $(\nu_0, \dots \nu_{n-1})$ of the same length, $|\#\{i : \omega_i = 1\} - \#\{i : \nu_i = 1\}| \leq 1$.
- The Rauzy fractal of a Sturmian word is an interval.

Notation

$$[0; a_1, a_2, a_3 \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Geometric representations of Sturmians



This geometric interpretation of s_θ is often called the staircase of s_θ.

Question What if we increase the number of letters?

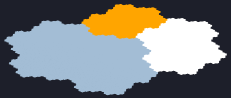
- A second prototypical model for a quasicrystal is a word generated by a substitution.
- Every Sturmian of slope θ, with θ a quadratic irrational, can be generated by a substitution.

Question What happens if we increase the number of letters?

The tribonacci substitution – $\varphi_{\text{trib}} : \{0,1,2\} \to \{0,1,2\}^*$

$$\varphi_{\text{trib}}(0) = (0,1) \quad \varphi_{\text{trib}}(1) = (0,2) \quad \varphi_{\text{trib}}(2) = (0)$$

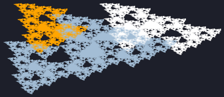
$$(0) \mapsto (0,1) \mapsto (0,1,0,2) \mapsto (0,1,0,2,0,1,0)$$
$$\mapsto (0,1,0,2,0,1,0,0,1,0,2,0,1)$$
$$\mapsto \ldots\ldots$$



The twisted tribonacci substitution – $\varphi_{\text{twtr}} : \{0,1,2\} \to \{0,1,2\}^*$

$$\varphi_{\text{twtr}}(0) = (1,0) \quad \varphi_{\text{twtr}}(1) = (0,2) \quad \varphi_{\text{twtr}}(2) = (0)$$

$$(0) \mapsto (1,0) \mapsto (0,2,1,0) \mapsto (1,0,0,0,2,1,0)$$
$$\mapsto (0,2,1,0,1,0,1,0,0,0,2,1,0)$$
$$\mapsto \ldots\ldots$$



Properties of φ_{trib} and φ_{twtr} Let φ be either φ_{trib} or φ_{twtr}

[Pytheas Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics*. Springer (2002)]

- φ has a power which admits a a substitution-fixed point.
- The orbit closure of each substitution-fixed point of φ is the same, and is minimal and aperiodic. The shift-space X_φ has zero topological entropy and is uniquely ergodic.
- The return times of occurrences of stings of length n in a substitution-fixed point of φ is $O(n)$.
- There exists a straight line ℓ in the direction of $\mathbf{R}$, such that the staircase of a substitution-fixed point of φ remains within a bounded distance from ℓ.
- The Rauzy fractal of two different substitution-fixed points of φ are translates of each other. Moreover, the Rauzy fractal of a substitution-fixed point of φ is the closure of its interior.

Question What happens if one stochastically mixes two substitutions?
Namely, what happens if we fix $p \in [0,1]$ and let $\varphi_p : \{0,1,2\} \to \{0,1,2\}^*$ be defined by

$$\varphi_p(0) = \begin{cases} (0,1) & \text{with probability } p \\ (1,0) & \text{with probability } 1-p \end{cases} \qquad \varphi_p(1) = (0,2) \qquad \varphi_p(2) = (0)$$

The random tribonacci substitution shift-space

- A word $u \in \{0,1,2\}^*$ is called legal if it appears as a subword of a realisation of $\varphi_p^n(a)$ for some $a \in \{0,1,2\}$ and $n \in \mathbb{N}_0$. We let $\mathcal{L}_{\text{r-trib}}$ denote the set of legal words.
- The random tribonacci substitution shift-space $X_{\text{r-trib}}$ is $\{\omega \in \{0,1,2\}^{\mathbb{N}_0} : \mathcal{L}(\omega) \subseteq \mathcal{L}_{\text{r-trib}}\}$.
- The substitution matrix of φ_p is $M_{\varphi_p} = (\mathbb{E}(\varphi_p(b)|_a))_{a,b\in\{0,1,2\}}$, namely

$$M_{\varphi_p} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{aligned} &\lambda = 1.83929\ldots \quad \lambda^3 - \lambda^2 - \lambda = 1 \\ &\mathbf{R} = (\lambda^2 + \lambda + 1)^{-1}(\lambda^2, \lambda, 1)^{\top} \end{aligned}$$

Properties of the random tribonacci substitution shift-space X_{φ_p}

[Rust and Spindeler. *Indag. Math.* **29** (2018)] and [Golhke. *Monatsh. Math.* **192** (2020)]

- The shift-space $X_{\text{r-trib}}$ has no periodic points, and has positive topological entropy.
- The shift-space $X_{\text{r-trib}}$ supports uncountably many ergodic measures. Namely, for $p \in [0,1]$ the frequency measure μ_p is ergodic, where for a fixed $a \in \{0,1,2\}$ and every $u \in \mathcal{L}_{\text{r-trib}}$ we set

$$\mu_p([u]) = \lim_{m\to\infty} \frac{\mathbb{E}(\varphi_p^m(a)|_u)}{\mathbb{E}(|\varphi_p^m(a)|)}.$$

Theorem [Gohlke, Mitchell, Rust and S. *Ann. Henri Poincaré* **24** (2023)]
We obtain explicit upper and lower bounds, which converge exponential, for $h(\mu_p)$.

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A construction for a canonical Rauzy fractal for the random tribonacci substitution
[Gohlke, Mitchell, Rust and S. arxiv.org/abs/2401.06732 (2024)]

- The following set is compact and contains a dense left-shift-orbit:
$$X^{\infty}_{\text{r-trib}} = \bigcap_{m \in \mathbb{N}_0} \{\nu \in \{0,1,2\}^{\mathbb{N}_0} : \nu \text{ is a realisation of } \varphi_p^m(\omega) \text{ for some } \omega \in X_{\text{r-trib}}\}$$
- We let $\mathbb{H}$ denote the hyperplane spanned by the non-dominant right eigenvectors of M_{φ_p}.
- We let π be the natural projection from $\mathbb{R}^3$ to $\mathbb{H}$ and for $u \in \mathcal{L}_{\text{r-trib}}$, set $\phi(u) = (u|_a)_{a \in \{0,1,2\}}$.
- The Rauzy fractal of the random tribonacci substitution corresponding to ω is defined to be
$$\mathcal{R}(\omega) = \pi(\{\phi(\omega_0, \omega_1 \ldots \omega_{n-1}) : n \in \mathbb{N}\}).$$

Theorem [Gohlke, Mitchell, Rust and S. arxiv.org/abs/2401.06732 (2024)]
If ω and $\nu \in X^{\infty}_{\text{r-trib}}$ have dense-left-shift orbits in $X_{\text{r-trib}}$, then we have the following.

- $\mathcal{R}(\omega) = \mathcal{R}(\nu)$
- $\mathcal{R}(\omega)$ is the closure of its interior
- $\mathcal{R}(\omega)$ is the attractor of a graph directed iterated function system

Theorem [Gohlke, Mitchell, Rust and S. arxiv.org/abs/2401.06732 (2024)]

For $\omega = (\omega_0, \omega_1, \dots) \in X_{\text{r-trib}}$ and $m \in \mathbb{N}$, we set

$$\nu_m(\omega) = \sum_{n=0}^{m} \delta_{\pi \circ \phi(\omega_0, \dots, \omega_n)}.$$

For μ_p-almost all $\omega \in X_{\text{r-trib}}$, the following weak limit exists.

$$\nu = \lim_{m \to \infty} m^{-1} \nu_m(\omega)$$