# Numeration 2024 Program

Utrecht, The Netherlands

3 – 7 June, 2024

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# Schedule

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- **10:00 – 11:00: Registration and coffee**
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- **12:00 – 12:30:** Hyperbolic multiscale tilings, partitions and numeration systems – Yotam Smilansky
- **12:30 – 14:00: Lunch**
- **14:00 – 14:30:** Analysis of Regular Sequences: Summatory Functions and Divide-and-Conquer Recurrences – Clemens Heuberger
- **14:30 – 15:00:** On the minima of $\alpha$-Brjuno functions – Ayreena Bakhtawar
- **15:00 – 15:30: Break**
- **15:30 – 16:30:** Mahler equations for Zeckendorf numeration – Reem Yassawi
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## Tuesday, 4 June

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- **11:30 – 12:00:** Numbers expressible by quotients or differences of two Pisot numbers – Artūras Dubickas
- **12:00 – 12:30:** Quadratic irrationals and their $N$-continued fraction expansions – Niels Langeveld
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  - Fiber denseness of intermediate $\beta$-shifts of finite type – Yun Sun
  - Non-decomposable quadratic forms over totally real fields – Magdaléna Tinková
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- **11:30 – 12:00:** Aperiodic sequences: Complexity and Rauzy fractals – Tony Samuel
- **12:00 – 12:30:** Geometry of restricted continued fraction digit sets and Lüroth digit sets – Sabrina Kombrink
- **12:30 – 14:00: Lunch**
- **14:00 – 14:30:** Optimal Representations of Gaussian and Eisenstein Integers using digit sets closed under multiplication – Adam Blažek, Edita Pelantová, Milena Svobodová
- **14:30 – 15:00:** On $\beta$-ary to binary conversion from an engineering point of view – Yutaka Jitsumatsu
- **15:00 – 15:30: Break**
- **15:30 – 16:30:** Periodic unique codings of fat Sierpinski gasket – Derong Kong

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- **11:30 – 12:00:** Weak Separation & Finite Type Property – Kevin Hare
- **12:00 – 12:30:** On numeration systems with positive and negative digits – Georges Grekos
- **12:30 – 14:00: Lunch**
- **14:00 – 14:30:** Tameness and amorphic complexity of automatic systems – Elżbieta Krawczyk
- **14:30 – 15:00:** Periodicity and pure periodicity in alternate base systems – Zuzana Masáková
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- **15:30 – 16:30:** Morphic sequences: characterization, visualization and equality – Hans Zantema

## Friday, 7 June

- **10:00 – 10:30:** Polynomials in Prime Ergodic Averages On Monothetic Groups – Radhakrishnan Nair
- **10:30 – 11:00:** Substitutive number systems – Paul Surer
- **11:00 – 11:30: Coffee break**
- **11:30 – 12:15:** Golden numeration systems – Michel Dekking
- **12:15 – 13:00:** Low discrepancy words and dynamical systems – Valérie Berthé

# Invited Speakers

**Eda Cesaratto**
**Universidad Nacional de General Sarmiento**

*Poisson genericity in numeration systems with exponentially mixing probabilities*

**Abstract:** We define Poisson genericity for infinite sequences in any finite or countable alphabet and with an invariant exponentially-mixing probability measure. A sequence is Poisson generic if the number of occurrences of blocks of symbols asymptotically follows a Poisson law as the block length increases. We prove that almost all sequences are Poisson generic. Our result generalizes Peres and Weiss' theorem about Poisson genericity of integer bases numeration systems and applies to exponentially mixing numeration systems. In particular, we obtain that for almost all real numbers their continued fraction expansions are Poisson generic.

Joint work with Nicolás Álvarez, Verónica Becher and Martín Mereb.

**Derong Kong**
**Chongqing University**

*Periodic unique codings of fat Sierpinski gasket*

**Abstract:** For $\beta > 1$ let $S_\beta$ be the Sierpinski gasket generated by the iterated function system

$$\left\{ f_{\alpha_0}(x,y) = \left(\frac{x}{\beta}, \frac{y}{\beta}\right), \quad f_{\alpha_1}(x,y) = \left(\frac{x+1}{\beta}, \frac{y}{\beta}\right), \quad f_{\alpha_2}(x,y) = \left(\frac{x}{\beta}, \frac{y+1}{\beta}\right) \right\}.$$

Then $O_\beta := \bigcup_{i \neq j} f_{\alpha_i}(\Delta_\beta) \cap f_{\alpha_j}(\Delta_\beta)$ is nonempty if and only if $1 < \beta \leq 2$, where $\Delta_\beta$ is the convex hull of $S_\beta$. In this talk we will discuss the periodic codings in the univoque set

$$\mathbf{U}_\beta := \left\{ (d_i)_{i=1}^\infty \in \{(0,0),(1,0),(0,1)\}^{\mathbb{N}} : \sum_{i=1}^{\infty} d_{n+i}\beta^{-i} \in S_\beta \setminus O_\beta \ \forall n \geq 0 \right\}.$$

More precisely, we will determine for each $k \in \mathbb{N}$ the smallest base $\beta_k \in (1,2]$ in which $\mathbf{U}_\beta$ contains a sequence of smallest period $k$ if and only if $\beta > \beta_k$. We show that each $\beta_k$ is a Perron number, and the sequence $(\beta_k)$ has infinitely many accumulation points. Furthermore, $\beta_{3k} > \beta_{3\ell}$ if and only if $k$ is larger than $\ell$ in the Sharkovskii ordering. This is joint work with Yuhan Zhang.

**Sébastien Labbé**
**CNRS & Université de Bordeaux**

*Numeration systems for aperiodic Wang tilings*

**Abstract:** In this talk, we aim to explain how the study of small sets of aperiodic Wang tiles and their associated nonperiodic tilings naturally leads to the theory of numeration systems.

In one-dimension, one-sided automatic sequences can be constructed by feeding the representation of nonnegative integers into an deterministic automaton with output. Similar results hold in two dimensions for describing configurations of the whole plane covered by Wang tiles. The statement of such results needs numeration systems representing all integers regarless of the sign, as done for example by the two's complement numeration system.

We present a complement version of Dumont-Thomas numeration systems for $\mathbb{Z}$ based on any two-sided periodic point with growing seed of a substitution. As such, we recover the two's complement numeration system and its Fibonacci analog. These numeration systems can be used to describe configurations in self-similar Wang subshifts. This includes the metallic mean Wang shifts introduced earlier this year whose

dynamical properties are related to the metallic mean numbers, that is, the positive root of $x^2 - nx - 1$ where $n \geq 1$ is an integer.

Surprisingly, another numeration system appear in the description of metallic mean Wang shifts, namely the balanced representation of real numbers which was already used by Kari and Culik in 1996 to show the existence of tilings with their aperiodic set of 14 and 13 tiles respectively.

Many of the results presented in the talk are joint work with Jana Lepšová who defended her Ph. D. thesis in May 2024.

**Joseph Vandehey**
**The University of Texas at Tyler**

*One digit, two digit, big digit, small digit*

**Abstract:** Our topic will be the theory of Iwasawa continued fractions, studied by the speaker and Anton Lukyanenko. This broad category includes several real and complex continued fraction algorithms, as well as higher-dimensional algorithms in the quaternions, octonions, and Heisenberg groups. Several important properties of real continued fractions extend to these higher-dimensional objects, including a connection to hyperbolic geometry. We will look at what is known about these Iwasawa continued fractions as well as what is still yet to be determined, focusing in particular on the question of perturbation: to what extent are dynamical properties preserved or destroyed by minute changes to the underlying algorithm, and how do these relate to the size of the continued fraction digits?

**Reem Yassawi**
**Queen Mary University of London**

*Mahler equations for Zeckendorf numeration*

**Abstract:** Let $U = (u_n)_{n \geq 0}$ be a Pisot numeration system. A sequence $(f_n)$ taking values over a commutative ring $R$, possibly infinite, is said to be $U$*-regular* if there exists a *weighted* automaton which outputs $f_n$ when it reads $(n)_U$. For base-$q$ numeration, with $q \in \mathbb{N}$, $q$-regular sequences were introduced and studied by Allouche and Shallit, and they are a generalisation of $q$-automatic sequences $(f_n)$, where $f_n$ is the output of a deterministic automaton when it reads $(n)_q$. Becker, and also Dumas, made the connection between $q$-regular sequences, and $q$*-Mahler type* equations. In particular a $q$-regular sequence gives a solution to an equation of $q$-Mahler type, and conversely, the solution of an *isolating*, or Becker, equation of $q$-Mahler type is $q$-regular.

We define generalised equations of Z-Mahler type, based on the Zeckendorf numeration system $Z$. We show that if a sequence over a commutative ring is Z-regular, then it is the sequence of coefficients of a series which is a solution of a Z-Mahler equation. Conversely, if the Z-Mahler equation is isolating, then its solutions define Z-regular sequences. We provide an example to show that there exist non-isolating Z-Mahler equations whose solutions do not define Z-regular sequences. Our proof yields a new construction of weighted automata that generate classical $q$-regular sequences.

This is joint work with Olivier Carton.

**Hans Zantema**
**Eindhoven University of Technology & Radboud University**

*Morphic sequences: characterization, visualization and equality*

**Abstract:** Morphic sequences form a natural class of infinite sequences, most times defined by fixed points of morphisms. They cover well-known examples like the Thue-Morse sequence and the Fibonacci sequence.

The recursive structure of such a morphic sequence is closely related to a dynamic radix enumeration system for the natural numbers.

In this talk we focus on the following three aspects of morphic sequences:

1. Equivalent characterizations of the class of morphic sequences. These include characterizations based on automata, by finiteness of a particular class of subsequences, and by rationality of infinite terms. Some of these extend similar well-known characterizations of automatic sequences.

2. Visualization by turtle graphics. Criteria have been developed resulting in either finite turtle figures with a lot of symmetry, or turtle figures with fractal patterns. Playing around with examples satisfying these criteria yield a wide range of amazing figures, each generated by a computer program of just a few lines.

3. Proving that different representations define the same morphic sequence. Surprisingly, exploiting a suitable criterion in many cases this can be done fully automatically by a simple computer program. This yields elementary induction proofs, however being quite complex by distinguishing several cases, while checking by hand is not needed as they are correct by construction.

All issues will be illustrated by numerous examples.

# General Audience Talks

**Valérie Berthé**
**CNRS**

*Low discrepancy words and dynamical systems*

**Abstract:** The chairman assignment problem can be stated as follows: $k$ states are assumed to form a union and each year a union chairman must be selected so that at any time the cumulative number of chairmen of each state is proportional to its weight. It is closely related to the (discrete) apportionment problem, which has its origins in the question of allocating seats in the house of representatives in the United States, in a proportional way to the population of each state. The richness of this problem lies in the fact that it can be reformulated both as a sequencing problem in operations research for optimal routing and scheduling, and as a symbolic discrepancy problem, in the field of word combinatorics, where the discrepancy measures the difference between the number of occurrences of a letter in a prefix of an infinite word and the expected value in terms of frequency of occurrence of this letter. We will see in this lecture how to construct infinite words with values in a finite alphabet having the smallest possible discrepancy, by revisiting a construction due to R. Tijdeman in terms of dynamical systems.

This is a collaborative work with O. Carton, N. Chevallier, W. Steiner, R. Yassawi.

**F. Michel Dekking**
**CWI Amsterdam & Delft University of Technology**

*Golden numeration systems*

**Abstract:** Let phi be the golden mean. There are two golden numeration systems. In the base phi numeration system, a natural number is written uniquely as a sum of powers of the golden mean with coefficients 0 and 1, where it is required that the product of two consecutive digits is always 0. The other system is the Zeckendorf numeration system. Here a natural number is written uniquely as a sum of Fibonacci numbers with coefficients 0 and 1, where it is required that the product of two consecutive digits is always 0. There are hundreds of papers on these systems. In this talk the focus will be on giving a precise description of the digit blocks that may occur in the expansions of the base phi numeration system. This yields in particular a new relationship between the two golden numeration systems.

# Contributed Talks

**Ayreena Bakhtawar**
**Scuola Normale Superior, Pisa (SNS)**

*On the minima of $\alpha$-Brjuno functions*

**Abstract:** An irrational number is called a Brjuno number if the sum of the series $\log(q_{n+1})/q_n$ converges, where $q_n$ is the denominator of the $n$-th principal convergent of the regular continued fraction. Brjuno numbers play an important role in the study of small divisors problems in dynamical systems. In 1988, J.-C. Yoccoz proved the optimality of the Brjuno condition for the linearization of quadratic polynomials, introducing a version of the Brjuno function well suited for estimate the size of Siegel disks. Motivated by the work of Balazard-Martin 2020, we study the scaling properties of the Brjuno function around its global and local minima. We give results both for the Brjuno function associated to the usual regular continued fraction expansion as well as for the generalized Brjuno function associated to $\alpha$-continued fractions where $\alpha \in [1/2, 1)$. This is based on joint work with Carlo Carminati and Stefano Marmi.

**Adam Blažek, Edita Pelantová, Milena Svobodová**
**Czech Technical University in Prague**

*Optimal Representations of Gaussian and Eisenstein Integers using digit sets closed under multiplication*

**Abstract:** We discuss an enumeration problem in two number systems, each one given by base $\beta \in \mathbb{C}$ and by set of digits $\mathcal{D} \subset \mathbb{C}$.

- Case 1: $\beta = \imath - 1$ and $\mathcal{D} = \{0, \pm 1, \pm \imath\}$,
- Case 2: $\beta = \omega - 1$ and $\mathcal{D} = \{0, \pm 1, \pm \omega, \pm \omega^2\}$, where $\omega = \exp(2\pi\imath/3)$.

The set $\left\{ \sum_{k=0}^{N-1} d_k \beta^k \,:\, N \in \mathbb{N}, d_k \in \mathcal{D} \right\}$ equals the ring of Gaussian integers $\mathbb{Z}[\imath]$ in Case 1, and the ring of Eisenstein integers $\mathbb{Z}[\omega]$ in Case 2.
Efficiency of multiplication algorithms in the two systems is guaranteed by three properties:

- Digit set $\mathcal{D}$ is closed under multiplication.
- Addition can be done by a $p$-local function, multiplication can be performed by on-line algorithms.
- In Case 1, any $x \in \mathbb{Z}[\imath]$ has the $w$-NAF representation (non-adjacent form) with $w = 3$, and this representation has the minimal Hamming weight among all representations of $x$. In Case 2, the same is true for any $x \in \mathbb{Z}[\omega]$ with $w = 2$.

We count the number $f(x)$ of optimal representations of $x \in \mathbb{Z}[\beta]$, i.e., representations of $x$ with the minimal Hamming weight. For any fixed $N \in \mathbb{N}$, we determine the maximal and the average value of $f(x)$, where $x$ belongs to $\mathcal{M}_N = \{x \in \mathbb{Z}[\beta] : \text{ length of } w\text{-NAF representation of } x \text{ is at most } N\}$.

**Christopher Cabezas**
**Université de Liège**

*Substitutive structures on general countable groups*

**Abstract:** Symbolic dynamics has been largely used to represent dynamical systems through a coding system. This method was initially developed by M. Morse and G. A. Hedlund. One commonly used coding method involves infinite sequences of morphisms defined on finitely generated monoids, known as directive sequences or $\mathcal{S}$-adic representations. Recent research has shown that understanding the underlying $\mathcal{S}$-adic structures of some subshifts is valuable for studying their dynamical properties. Considering the previous studies and acknowledging the effectiveness of the $\mathcal{S}$-adic framework, it is natural to ask whether this setting is useful beyond the one-dimensional case. In 2023, the notion of constant-shape substitutions was introduced, as a multidimensional analogue for constant-length substitutions, marking a first attempt to study multidimensional substitutions in a broader class than those defined solely by rectangular and square supports. In this talk we are going to discuss a way to define in a general way an analogue of constant-shape substitutions for general countable group actions. The geometry of these groups is an important feature to consider. We are going to present some new dynamical properties about them, and discuss how to obtain other representations. This is a joint work with N. Bitar and P. Guillon.

**Artūras Dubickas**
**Vilnius University**

*Numbers expressible by quotients or differences of two Pisot numbers*

**Abstract:** In 1945, Salem himself proved that every Salem number is expressible as a quotient of two Pisot numbers. On the other hand, in 2004 the author showed that every positive algebraic number is a quotient of two Mahler measures. Recall that the *Mahler measure* $M(\alpha)$ of a nonzero algebraic number $\alpha$ is the modulus of the product of its conjugates lying outside the unit circle and the leading coefficient of its minimal polynomial in $\mathbb{Z}[x]$. Hence, for a real algebraic number $\alpha > 1$, we have $M(\alpha) = \alpha$ if and only if $\alpha$ is a Salem number or a Pisot number. The following theorem implies both these results:

**Theorem 1.** *Every real positive algebraic number $\alpha$ of degree $d$ is expressible as a quotient of two Pisot numbers of degree $d$ from the field $\mathbb{Q}(\alpha)$.*

Earlier, the author also investigated various sumsets and difference sets involving Salem and Pisot numbers. Now, we show that

**Theorem 2.** *Every Salem number is expressible as a difference of two Pisot numbers*

and study which other algebraic integers are expressible in this way.

**Georges Grekos**
**Université de Saint-Étienne**

*On numeration systems with positive and negative digits*

**Abstract:** Work in progress and in common with Rita GIULIANO (Pisa) and Labib HADDAD (Paris).

The numeration systems considered in this talk have as *basis* either the powers $\{k^0, k^1, k^2, k^3, \dots\}$ of an integer $k \geq 2$ or, more generally, the so called *Cantor* (or *sterling* !) basis $\{b_0 = 1, b_1 = k_1, \dots, b_{i+1} = b_i k_{i+1}, \dots\}$ where $(k_i)_{i \geq 1}$ is a given sequence of integers with $k_i \geq 2$ for all $i \geq 1$. The *digits* $c_i$ will be integers subject to certain conditions. The system will be *complete* and *non redundant*: every integer $n \in \mathbf{Z}$ will have a unique representation $n = \sum_{i=0}^{\infty} c_i b_i$ where all, except finitely many, $c_i$ are zero, each $c_i \in \mathbf{Z}$ depends on $k_{i+1}$ and is uniquely determined as for its sign and for its absolute value $|c_i|$. We prove:

**Theorem 1.-** Given a Cantor basis $(b_i)_{i \geq 0}$, every nonzero integer $n$ has a unique representation of the form $n = -c_1 b_{m_1} + c_2 b_{m_2} - \cdots + (-1)^j c_j b_{m_j} + \cdots + (-1)^s c_s b_{m_s}$ where $1 \leq c_j < k_{m_j+1}$ for every index $j$ and the $s$ integers $m_j$ form a strictly increasing sequence $0 \leq m_1 < m_2 < \cdots < m_s$.

Notation: Let $a > 0$ be an integer. We define a set of *residues*

$R(a) = \{-a/2, -a/2+1, \dots, -1, 0, 1, \dots, a/2\}$ if $a$ is even;

$R(a) = \{-(a-1)/2, \ldots, -1, 0, 1, \ldots, (a+1)/2\}$ if $a$ is odd.

**Theorem 2.-** Given a Cantor basis $(b_i)_{i\geq 0}$, every nonzero integer $n$ has a unique representation of the form $n = c_0 b_0 + c_1 b_1 + \cdots + c_s b_s$ where, for every $i$, $0 \leq i \leq s$, $c_i \in R(k_{i+1})$, $c_s \neq 0$, and the $c_i$'s are subject to the following condition: If one of the $c_i$'s with $i < s$ is an extreme point of the interval $R(k_{i+1})$, then $c_i c_{i+1} \geq 0$ and $c_{i+1}$ is not an extreme point of the interval $R(k_{i+2})$.

The end of the talk will be devoted to some open questions concerning the cost function $C(n) = \sum_{i=0}^{\infty} |c_i|$.

**Kevin G. Hare**
**University of Waterloo**

*Weak Separation & Finite Type Property*

**Abstract:** Consider the set of numbers $K_a$ representable by $K_a = \{\sum_{i=1}^{\infty} \frac{b_i}{7^i} : b_i \in \{0, a, 6\}$ for some $a \in (0,6)$. We easily see that $K_a \subsetneq [0,1]$. For $a \in (1,5)$ we see that every $x \in K_a$ has a unique representation. If $a = 1$ or $a = 5$ then almost all numbers have a unique representation. Those that do not have a unique representation have exactly two representations, and both representations are periodic. We present an example of an $a$ such that where almost all numbers have a unique representation, and for those numbers with multiple representations, then these representations are necessarily aperiodic. This provides a partial answer to the relationship between weak separation property and convex finite type condition for iterated function systems.

**Clemens Heuberger**
**University of Klagenfurt**

*Analysis of Regular Sequences: Summatory Functions and Divide-and-Conquer Recurrences*

**Abstract:** In simplest terms, a sequence $x$ is called $q$-regular for some integer $q \geq 2$ if there are square matrices $A_0, \ldots, A_{q-1}$, a row vector $u$ and a column vector $w$ such that for all integers $n \geq 0$,

$$x(n) = uA_{n_0} \ldots A_{n_{\ell-1}} w$$

where $(n_{\ell-1}, \ldots, n_0)$ is the $q$ary expansion of $n$.

In the asymptotic analysis of regular sequences, it is usually advisable to study their summatory function because the original sequence has a too fluctuating behaviour. It turns out that the summatory function $N \mapsto \sum_{0\leq n<N} x(n)$ of a $q$-regular sequence $x$ has an asymptotic expansion

$$\sum_{0\leq n<N} x(n) = \sum_{\substack{\lambda\in\sigma(C)\\ |\lambda|>R}} N^{\log_q \lambda} \sum_{0\leq k<m_C(\lambda)} \frac{(\log N)^k}{k!}\, \Phi_{\lambda k}(\log_q N) + O\bigl(N^{\log_q R}(\log N)^{\kappa}\bigr)$$

as $N \to \infty$, where the $\Phi_{\lambda k}$ are suitable 1-periodic continuous functions and $\sigma(C)$, $m_C$, $R$, $\kappa$ depend on the regular sequence.

It might be that the process of taking the summatory function has to be repeated if the sequence is fluctuating too much. In this talk we report on results that for all regular sequences except for some degenerate cases, repeating this process finitely many times leads to a "nice" asymptotic expansion containing periodic fluctuations whose Fourier coefficients can be computed using the results on the asymptotics of the summatory function of regular sequences.

In a recent paper, Hwang, Janson, and Tsai perform a thorough investigation of divide-and-conquer recurrences. These can be seen as 2-regular sequences. By considering them as the summatory function of their forward difference, the results on the asymptotics of the summatory function of regular sequences become applicable. We thoroughly investigate the case of a polynomial toll function.

Based on joint work with Daniel Krenn and Tobias Lechner.

**Yutaka Jitsumatsu**
**Kyushu University**

*On $\beta$-ary to binary conversion from an engineering point of view*

**Abstract:** A $\beta$ encoder (Daubechise et al., 2006) is an analog-to-digital (AD) converter based on $\beta$ transformation. This AD converter was developed to overcome the drawback that AD conversion methods based on binary expansion are not robust to threshold variations. The goal of a $\beta$ encoder is to obtain coefficients of $\beta$-expansion of the input analog value $x$ with $\beta \in (1, 2]$. A scale-adjusted $\beta$ expansion is given by

$$x = (\beta - 1) \sum_{i=1}^{\infty} a_i \beta^{-i}.$$

There are uncountably many $\beta$ expansions for a single $x$. Let $\nu_i$ denotes the threshold at the $i$-th iteration, allowing for fluctuations. We can model the process of $\beta$ encoder as follows: With initial value $x_0 = x$,

$$a_i = Q_{\nu_i}(\beta x_{i-1}), \quad x_i = \beta x_{i-1} - a_i, \quad i \geq 1$$

where $Q_\nu(x) = 0$ if $x < \nu$ and $Q_\nu(x) = 1$ if $x \geq \nu$. If $\nu_i \in [1, 1/(\beta - 1)]$ is satisfied, the $n$-bit approximation error $|x - (\beta - 1) \sum_{i=1}^{n} a_i \beta^{-i}|$ decreases exponentially in $n$. Hence $\beta$ encoder is robust to the fluctuation of the threshold.

The $\beta$-ary to binary conversion (Matsumura and Jitsumatsu, 2016) is a post-processing for a $\beta$ encoder, which generates the binary expansion $b_j$'s of $x$ whose scale-adjusted $\beta$ expansion is $a_i$. Our central concern is how many bits of $\beta$ expansion are needed to correctly determine the first $n$ binary expansions of $x$.

In this talk we discuss i) the approximation error of the proposed method, ii) the effect of mismatches in $\beta$ values, and iii) the extension to the case $\beta > 2$.

**Sabrina Kombrink**
**University of Birmingham**

*Geometry of restricted continued fraction digit sets and Lüroth digit sets*

**Abstract:** In this talk we will discuss geometric features of limit sets of infinitely generated conformal graph directed systems. Examples of such limit sets, which we will focus on, are the restricted continued fraction digit sets $F_\Lambda := \{[a_1, a_2, \ldots] \ : \ a_n \in \Lambda, \ n \in \mathbb{N}\}$ for given $\Lambda \subset \mathbb{N}$ and analogously defined restricted Lüroth digit sets. Here, $[a_1, a_2, \ldots]$ denotes the continued fraction expansion with digits $a_1, a_2, \ldots$. This is joint work with M. Kesseböhmer.

**Elżbieta Krawczyk**
**Jagiellonian University**

*Tameness and amorphic complexity of automatic systems*

**Abstract:** Amorphic complexity–introduced by Fuhrmann, Gröger, and Jäger–is a relatively new invariant of topological dynamical systems useful in the study of aperiodic order and low complexity dynamics. Tameness is a well-studied notion usually defined in terms of the size of the Ellis semigroup of the system.

In our work we study amorphic complexity and tameness in the class of automatic systems–systems arising from constant length substitutions. We provide a closed formula for the amorphic complexity of any minimal automatic system and show that tameness of such systems can be succinctly characterized through amorphic complexity: A minimal automatic system is tame if and only if its amorphic complexity is zero (in which case the system is finite) or one.

Our proofs use methods from fractal geometry and introduce some new dynamically-defined pseudometrics. These methods seem suitable for study of nonminimal automatic systems as well as other systems of S-adic nature. Time permitting we will touch on some possible generalisations in these directions.

Joint work with Maik Gröger.

**Niels Langeveld**
**Montan University Leoben**

*Quadratic irrationals and their $N$-continued fraction expansions*

**Abstract:** A real number has a periodic regular continued fraction expansion if and only if it is a quadratic irrational. This classical result is also true for many other continued fraction families. Is it also true for $N$-continued fractions expansions? These are expansions of the form:

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{\ddots}}} \tag{1}$$

where $N \in \mathbb{N}_{>1}$ and $d_i \in \mathbb{N}$. $N$-continued fractions and regular continued fractions differ in many ways. For example, $N$-continued fractions are not unique. In this talk we focus on the following question. For a quadratic irrational $x$, what can we say about the periodicity of its $N$-continued fraction expansions?

To find continued fractions of the form (1) we can use dynamical systems. Here we will focus on the following family, for which we show that for certain choices of $\alpha$ there exist quadratic irrationals with an a-periodic expansion. Let $\alpha \in (0, \sqrt{N} - 1]$, and define the map $T_{N,\alpha} : [\alpha, \alpha + 1] \to [\alpha, \alpha + 1)$ as

$$T_{N,\alpha}(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} - \alpha \right\rfloor.$$

The map $T_{N,\alpha}$ generates $N$-continued fractions with only finitely many different digits. Moreover, $T_{N,\alpha}$ maps quadratic irrationals onto quadratic irrationals. Let $x_0 \in [\alpha, \alpha + 1]$ be a quadratic irrational that satisfies $A_0x^2 + B_0x + C_0 = 0$ and define $x_n = T^n_{N,\alpha}(x_0)$ which satisfies $A_nx^2 + B_nx + C_n = 0$ for some $A_n, B_n, C_n \in \mathbb{Z}$. For the determinant of these equations we can show that

$$B_n^2 - 4A_nC_n = N^{2n}(B_0^2 - 4A_0C_0) \tag{2}$$

holds. Now the idea is to pick $\alpha$ and initial coefficients $A_0, B_0$ and $C_0$ such that $A_n, B_n$ and $C_n$ are co-prime for all $n$. In this way (2) gives us the a-periodicity. This talk is joint work with Cor Kraaikamp.

**Zuzana Masáková**
**Czech Technical University in Prague**

*Periodicity and pure periodicity in alternate base systems*

**Abstract:** Alternate base $\boldsymbol{\mathcal{B}}$ is given by a $p$-tuple $(\beta_1, \beta_2, \ldots, \beta_p)$ of real numbers greater than 1. We investigate in which cases all rational numbers $\frac{p}{q}$ in the interval $(0, 1)$ have an eventually periodic $\boldsymbol{\mathcal{B}}$-expansion. We show that this property forces the product $\delta = \beta_1\beta_2 \cdots \beta_p$ to be a Pisot or a Salem number. Analogic conclusion was earlier derived by Charlier, Cisternino and Kreczman, under a stronger requirement that $\frac{p}{q}$ has an eventually periodic expansion in every alternate base obtained by a cyclic shift of the original $p$-tuple.

We further examine under which circumstances there exists a $\gamma > 0$ such that every rational number in the interval $(0, \gamma)$ has a purely periodic $\boldsymbol{\mathcal{B}}$-expansion. We show that a necessary condition for this phenomenon is that $\delta$ is a Pisot or a Salem unit. We also provide a sufficient condition. We thus generalize the results

known for the Rényi numeration system, i.e. for the case when $p = 1$, obtained by Schmidt, Akiyama, Adamczewski et al. and others. At the end, we present a class of alternate bases with $p = 2$, for which $\gamma$ can be chosen to be 1.

Joint work with Edita Pelantová.

**Radhakrishnan Nair**
**University of Liverpool**

*Polynomials in Prime Ergodic Averages On Monothetic Groups*

**Abstract:** We say a topological group $G$, is monothetic if it contains an element $\alpha$, called a generator, such that the closure of $(n\alpha)_{n=1}^{\infty}$ is $G$. Monothetic groups are of necessity, abelian. For $\alpha_1, \dots, \alpha_k \in G$, let

$$\rho(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \dots + \alpha_1 n + \alpha_0.$$

Also let $(p_n)_{n\geq 1}$ be the sequence of rational primes and let $\lambda$ denote Haar measure on $G$.

Suppose $G$ is a compact monothetic group, where one of the elements $\alpha_1, \dots, \alpha_k \in G$ is a generator and that $f \in L^p(G)$ for $p > 1$. We describe the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + \rho(p_n)),$$

for almost all $x$ with respect to $\lambda$ in terms of the connectivity and arithmetic character of $G$.

This is joint work with J. Hancl (Ostrava) and J.L. Verger-Gaugry (Chembery/Grenoble)

**Eric Rowland**
**Hofstra University**

*Base-$\frac{p}{Q}$ structure of states in automata arising from Christol's theorem*

**Abstract:** A celebrated theorem of Christol states that a sequence $a(n)_{n\geq 0}$ of elements in the finite field $\mathbb{F}_q$ is algebraic if and only if it is $q$-automatic. One direction of Christol's theorem was generalized by Denef and Lipshitz as follows. If a sequence $a(n)_{n\geq 0}$ of integers is algebraic then, for every $\alpha \geq 1$, the sequence $(a(n) \bmod p^{\alpha})_{n\geq 0}$ is $p$-automatic. We are interested in the size of the minimal automaton for this sequence (measured as the number of states). Experiments suggest that the size grows exponentially as a function of $\alpha$, but the bound obtained from the construction is doubly exponential. To reconcile this discrepancy, we describe a new base-$\frac{p}{Q}$ numeration system, where $Q \in (\mathbb{Z}/p^{\alpha}\mathbb{Z})[x, y]$ and where the digits belong to $(\mathbb{Z}/p\mathbb{Z})[x, y]$. We show that each state in the automaton has a unique representation in this numeration system and, moreover, that the degrees of the digits are bounded. This restricted structure allows us to obtain a singly exponential bound on the number of states.

Joint work with Reem Yassawi.

**Tony Samuel**
**University of Birmingham**

*Aperiodic sequences: Complexity and Rauzy fractals*

**Abstract:** Aperiodic sequences and sequence spaces form prototypical mathematical models of quasicrystals. The most quintessential examples include subshifts of Sturmian words and substitutions, which are ubiquitous objects in ergodic theory and aperiodic order. Two of the most striking features these shift

spaces have, are that they have zero topological entropy and are uniquely ergodic. Random substitutions are a generalisation of deterministic substitutions, and in stark contrast to their deterministic counterparts, subshifts of random substitutions often have positive topological entropy and exhibit uncountably many ergodic measures. Moreover, they have been shown to provide mathematical models for physical quasicrystals with defects.

We will begin by talking about subshifts generated by Sturmian words and ways to measure their complexity beyond topological entropy, and show how this measure of complexity can be used to build a classification via Jarník sets. We will then build a bridge between these subshifts and subshifts of random substitutions. We will conclude with some recent dynamical results on subshifts of random substitutions and ways to visualise these subshifts. Namely, we will present a method to build a new class of Rauzy fractals.

**Yotam Smilansky**
**The University of Manchester**

*Hyperbolic multiscale tilings, partitions and numeration systems*

**Abstract:** Substitution rules provide a classical method for constructing aperiodic tilings via a substitution-inflation procedure. When distinct incommensurable scales are allowed in the substitution rule a different approach is required, and new geometric objects emerge. In my talk I will introduce multiscale substitution tilings and their hyperbolic liftings into the upper-half space $\mathbb{H}^{d+1}$, which may be viewed as extensions of constructions previously considered by, among others, Penrose, Kakutani and Kamae, who studied related objects in the context of numeration systems, and essentially illustrated by Escher. I will then describe recent results about such tilings and about the geodesic and horospheric actions on the associated tiling spaces, including a prime orbit theorem for the geodesic flow. Based on joint work with Yaar Solomon.

**Paul Surer**
**Universität für Bodenkultur**

*Substitutive number systems*

**Abstract:** In the presentation we show how we can associate a primitive substitutions with a family of non-integer positional number systems with respect to the same base but with different sets of digits. In this way we generalise the classical Dumont-Thomas numeration which corresponds to one special case. Therefore, our concept also covers beta-expansions induced by Parry numbers. But we also establish links to variants of beta-expansions such as symmetric beta-expansions. In other words, we unify several well-known notions of non-integer representations within one general framework. A focus is set on finiteness and periodicity properties. We will see that these characteristics mainly depend on the substitution. This observation allows us to relate several known notions of finiteness properties.

# Pitch and Poster Session

**Sohail Farhangi**
**University of Adam Mickiewicz**

*Cantor series for which normality and distribution normality coincide*

**Abstract:** A natural generalization of the notion of base $b$ expansions is to fix a **basic sequence** $Q = (q_n)_{n=1}^{\infty}$ with $q_n \geq 2$, and consider the[1] base $Q$ expansion of $y \in [0,1]$ given by

$$y = \frac{a_1}{q_1} + \frac{a_2}{q_1 q_2} + \frac{a_3}{q_1 q_2 q_3} + \cdots = \sum_{n=1}^{\infty} \frac{a_n}{\prod_{i=1}^{n} q_i} = 0.a_1 a_2 \cdots a_n \cdots_Q\,, \tag{3}$$

with $0 \leq a_i < q_i$. The base $b$ expansions correspond to the case in which $Q$ is the constant sequence $(b)_{n=1}^{\infty}$. There are two natural notions of normality that we can associate to a basic sequence. We say that $y \in [0,1]$ is **$Q$-normal** if every block of digits appearing in the base $Q$ expansion of $y$ appears with the correct frequency (which we do not define precisely here), and we denote the set of such $y$ by $\mathcal{N}(Q)$. We say that $y \in [0,1]$ is **$Q$-distribution normal** if the sequence $(x \prod_{i=1}^{n} q_n)_{n=1}^{\infty}$ is uniformly distributed in $[0,1]$, and we denote the class of such $y$ by $\mathcal{DN}(Q)$. In contrast to the situation of base $b$, there exists basic sequences for which $\mathcal{N}(Q) \neq \mathcal{DN}(Q)$. It is therefore natural to try and find classes of $Q$ for which $\mathcal{N}(Q) = \mathcal{DN}(Q)$, which is the goal of the present work. We remark that this is only the beginning of a larger program in which we try to find $Q$ for which the theory of base $b$ normality extends to a theory of normality base $Q$.

Joint work with Bill Mance.

**Peej Ingarfield**
**University of Manchester**

*Thermodynamic Formalism of Self Similar Overlapping Measures*

**Abstract:** A key question in Fractal geometry is to understand the Hausdorff dimension of sets or measures. In recent years there has been progress made to understand more complicated fractal structures, in particular overlapping self-similar measures. There has been focus on the study of Bernoulli convolutions, which lead to overlapping fractals with a variable contraction rate. Rather than studying these we shall be looking at orthogonal projections of the uniform measure on the Sierpinski triangle. These projections form overlapping self similar sets with a variable translation parameter. The main result of this work is to make a connection between dimension theory of these IFSs and thermodynamic formalism of the doubling map restricted to rational slices of the torus. Of note is how we establish a correspondence between the varying transnational parameter and varying rational slices.

**Pascal Jelinek**
**Montanuniversität Leoben**

*Collisions of digit sums in two bases*

**Abstract:** The problem of the infinitude of numbers that have the same sum of digits in multiple coprime bases, we call such numbers collisions, has been open for many decades. Only recently, in 2023, Spiegelhofer managed to prove a lower bound on collisions in bases 2 and 3, and hence showed for the first time that there are infinitely many such numbers. His methods rely on the following two properties of the sum of digits function in arithmetic progressions of step size $d < N^{1/2-\varepsilon}$.

Firstly the sum of digits of these numbers needs to be concentrated around the expected value and secondly

[1]The expansion is called a Cantor series and is unique for all but countably many $y \in [0,1]$.

the sum of digits needs to be equidistributed modulo $m$, where $m \leq \sqrt{\log(N)/\log(\log(N))}$.
In this talk, we will show how both properties can be derived for $d$ of size at most $N^{1-\varepsilon}$, under some mild assumptions on $d$. Using this, we can find a lower bound on collisions in coprime bases $p$ and $q$, and hence show that there are infinitely many such numbers.

**Jakub Krásenský**
**Czech Technical University in Prague**

*Number systems in imaginary quadratic fields*

**Abstract:** We study a certain type of generalised number systems which allow unique representation of algebraic integers in a given number field: For a ring $R$, take any $\beta \in R$ and let $D \ni 0$ be a finite subset of $R$. We say that $(\beta, D)$ is a *GNS* in $R$ if every nonzero element of $R$ has a unique representation of the form

$$x = \sum_{k=0}^{N} \beta^k a_k, \qquad \text{where } N \in \mathbb{N}_0,\ a_k \in D,\ a_N \neq 0.$$

The element $\beta$ is called *radix* or *base* and $D$ is the *alphabet*. It is clear that the alphabet must be a full residue system modulo $\beta$.

GNSs in imaginary quadratic fields have been studied among others by W. Penney, I. Kátai, J. Szabó, G. Steidl, W. Gilbert or A. Vince. By results of Imre Kátai, it is known that if $R = \mathcal{O}_K$ is the ring of integers in an imaginary quadratic number field $K$, then $\beta \in \mathcal{O}_K$ can serve as a radix of a GNS precisely if $|\beta| \neq 1$ and $|1-\beta| \neq 1$. We extend this result also to all non-maximal imaginary quadratic orders (such as $\mathbb{Z}[\sqrt{3}\mathrm{i}]$) and, furthermore, characterise when it is possible to construct infinitely many GNSs with given radix $\beta$. This is joint work with A. Kovács.

**Savinien Kreczman**
**Université de Liège**

*Confluent alternate numeration systems*

**Abstract:** We study normalisation as a rewriting system in the framework of two-way alternate bases.

In this framework, nonnegative real numbers are represented by right-infinite words with a decimal point. A number may have multiple representations, among which one is distinguished unsing a greedy algorithm and called the expansion. The problem of normalisation is to find, given a word, the expansion of the number it represents.

We study this problem by seeing normalisation as a rewriting system, taking a representation as input and iteratively rewriting factors which cannot appear in expansions while keeping the value of the word constant, until the expansion is reached.

We are especially interested in the numeration systems for which this associated rewriting system is confluent, that is, two words that can be obtained by rewriting a common start word can in turn be rewritten as a common end word. We obtain a characterisation of those systems: up to technicalities, all but the last digit in any expansion of 1 must be maximal.

A connection is made to the problem of the equality between the spectrum and the integers of a numeration system. This problem asks whether all numbers that admit a representation only to the left of the fractional point also have their expansion only to the left of the fractional point. Using a similar framework of rewriting rules, we find a class of systems where this equality is reached, with a criterion similar in statement to the one mentioned above.

Joint work with Émilie Charlier, Zuzana Masáková and Edita Pelantová.

**Renan Laureti**
**Université de Lorraine**

*Dependances between the digits of $\beta$-expansions*

**Abstract:** A common process in the study of normality to a given base is to consider the digits of the expansion of a real number in that base as random variables, in order to use results of the theory of probability. For example, one way to prove the theorem of Borel, which states that almost every real number (for Lebesgue measure) is absolutely normal, that is normal simultaneously to every integer base, is to use the law of large numbers. Another result that is used in effective algorithms of construction of normal numbers is Hoeffding's inequality, that allows to bound the amount of non $(\varepsilon, k)$-normal numbers in a given base. However, these results can only be applied to bases where the random variables representing the digits are independant. This is not the case for $\beta$-expansions, and in this poster we will see more about the dependances of digits in those bases. We will then see how we can still apply versions of Hoeffding's inequality in some cases using a structure of Markov's chain of the random variables representing the $\beta$-expansions digits when they differentiated between the types of cylinders associated to those digits.

**Yao-Qiang Li**
**Guangdong University of Technology**

*Expansions of generalized Thue-Morse numbers*

**Abstract:** We introduce generalized Thue-Morse numbers of the form

$$\pi_\beta(\theta) := \sum_{n=1}^{\infty} \frac{\theta_n}{\beta^n}$$

where $\beta \in (1, m+1]$ with $m \in \mathbb{N}$ and $\theta = (\theta_n)_{n\geq 1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ is a generalized Thue-Morse sequence previously studied by many authors in different terms. This is a natural generalization of the classical Thue-Morse number $\sum_{n=1}^{\infty} \frac{t_n}{2^n}$ where $(t_n)_{n\geq 0}$ is the well-known Thue-Morse sequence $01101001\cdots$. We study when $\theta$ would be the unique, greedy, lazy, quasi-greedy and quasi-lazy $\beta$-expansions of $\pi_\beta(\theta)$, and generalize a result given by Kong and Li in 2015. In particular we deduce that the shifted Thue-Morse sequence $(t_n)_{n\geq 1}$ is the unique $\beta$-expansion of $\sum_{n=1}^{\infty} \frac{t_n}{\beta^n}$ if and only if it is the greedy expansion, if and only if it is the lazy expansion, if and only if it is the quasi-greedy expansion, if and only if it is the quasi-lazy expansion, and if and only if $\beta$ is no less than the Komornik-Loreti constant.

**Yun Sun**
**South China University of Technology**

*Fiber denseness of intermediate $\beta$-shifts of finite type*

**Abstract:** We focus on $T_{\beta,\alpha}(x) = \beta x + \alpha \pmod 1$, $x \in [0, 1]$ and $(\beta, \alpha) \in \Delta := \{(\beta, \alpha) \in \mathbf{R}^2 : \beta \in (1, 2) \text{ and } 0 < \alpha < 2 - \beta\}$. The $T^{\pm}_{\beta,\alpha}$-expansions $\tau^{\pm}_{\beta,\alpha}(x)$ of critical point $c_{\beta,\alpha} = \frac{1-\alpha}{\beta}$ are denoted as $(k_+, k_-)$. Let $\Delta(k_+) := \{(\beta, \alpha) \in \Delta : \tau^{+}_{\beta,\alpha}(c_{\beta,\alpha}) = k_+\}$ with $k_+$ being periodic, we state that $\Delta(k_+)$ is a smooth curve which can be regarded as a fiber. We extend the results of Parry (1960) and show that, the set of $(\beta, \alpha)$ with its $\Omega_{\beta,\alpha}$ being a SFT is dense in $\Delta(k_+)$. Similarly for the fiber $\Delta(k_-)$.

When considering another fiber $\Delta(\beta) := \{(\beta, \alpha) \in \Delta : \beta \in (1, 2) \text{ is fixed}\}$, we demonstrate that when $\beta$ is not a multinacci number, there are only countably many distinct matching intervals on $\Delta(\beta)$. We prove that the set of $(\beta, \alpha)$ with $\Omega_{\beta,\alpha}$ being a SFT is dense in each matching interval.

Joint work with Bing Li and Yiming Ding.

**Magdaléna Tinková**
**Czech Technical University in Prague**

*Non-decomposable quadratic forms over totally real fields*

**Abstract:** Let $K$ be a totally real field. In this talk, we will consider $n$-ary quadratic forms $Q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ where coefficients $a_{ij}$ belong to the ring of algebraic integers $\mathcal{O}_K$ of $K$. We say that $Q$ is totally positive semi-definite if $Q(\gamma_1, \ldots, \gamma_n) \in \mathcal{O}_K^+ \cup \{0\}$ for all $\gamma_i \in \mathcal{O}_K$, where $\mathcal{O}_K^+$ denotes the set of totally positive elements in $\mathcal{O}_K$, i.e., of those elements whose all conjugates are positive. Moreover, our quadratic form is non-decomposable if it cannot be written as $Q = Q_1 + Q_2$ where $Q_1$ and $Q_2$ are totally positive semi-definite quadratic forms.

So far, the research on non-decomposable quadratic forms mostly focused on the case when $K = \mathbb{Q}$. For general totally real fields, Baeza and Icaza found a constant $C_{BI}$ satisfying the following. If the algebraic norm of the determinant $N(\det(Q))$ of $Q$ is greater than $C_{BI}$, then $Q$ can be written as $Q = L^2 + H$ where $L$ is a linear form in $n$ variables and $H$ is a totally positive definite form, which implies that $Q$ is decomposable. In this talk, we will show that their result can be improved considering decompositions of the form $Q = \alpha L^2 + H$ where $\alpha \in \mathcal{O}_K$ is totally positive. Moreover, we will discuss some related results.

Joint work with Pavlo Yatsyna.

**Nathan Toumi**
**IECL Lorraine**

*The level of distribution of the sum-of-digits function in arithmetic progressions*

**Abstract:** The aim of our work is to generalize, for any base of numeration and a more general sequence than the Thue–Morse sequence, recent results obtained by L. Spiegelhoferregarding the level of distribution of sequences related to the sum-of-digits function. He showed that the Thue–Morse sequence has level of distribution of 1 improving on a former result of Fouvry and Mauduit. The heart of the proof for our generalization lies in an explicit estimate on Gowers norms, it follows the lines of Konieczny's proof. We make the calculations explicit to provide an explicit gain in the exponent in the main results.

**Hichem Zouari**
**Institut Élie Cartan de Lorraine (IECL)**

*Mean values of some additive arithmetical functions over friable integers*

**Abstract:** Let $S(x, y)$ be the set of integers up to $x$, all of whose prime factors are $\leq y$, and $s_q(n)$ be the sum-of-digits function in base $(q \geq 2)$ of the positive integer $n$. Our main result is to estimate the sum $\sum_{n \in S(x,y)} v(n)$, where $v(n)$ is either $\tilde{\omega}(n)$ or $\tilde{\Omega}(n)$, the number of distinct prime factors and the total number of prime factors $p$ of a positive integer $n$, such that $s_q(p) \equiv a \mod b$, $(a, b \in \mathbb{Z})$.