Periodicity and pure periodicity in alternate bases

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(joint work with Edita Pelantová)

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The **greedy** representation of $x \in [0,1)$ in the system $\mathcal{B}$ is the **lexicographically greatest** sequence $\boldsymbol{d(\mathcal{B}, x)} = x_1 x_2 x_3 \cdots$ such that

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Rényi	$\beta_k = \beta \in \mathbb{R}$	shifts	$\sigma^j(\mathcal{B}) = (\beta_{i+1}, \ldots, \beta_{i+p})^\omega$

Periodicity

Alternate base $\mathcal{B} = (\beta_1, \ldots, \beta_p)$, $\delta = \prod_{i=1}^{p} \beta_i$

$$\mathbf{Per}(\mathcal{B}) = \left\{ x \in [0,1) : d(\mathcal{B}, x) \text{ is eventually periodic} \right\}$$

$\mathrm{Per}(\mathcal{B}) \subset \mathbb{Q}(\beta_1, \ldots, \beta_p)$.

Theorem (Schmidt 1980)

Let $\beta > 1$.

1. *If $\mathbb{Q} \cap [0,1) \subset \mathrm{Per}(\beta)$, then β is Pisot or Salem.*
2. *If β is Pisot, then $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1)$.*

Periodicity in alternate bases

Theorem (Charlier, Cisternino & Kreczman 2022)

Let $\mathcal{B} = (\beta_1, \beta_2, \ldots, \beta_p)$ *be an alternate base,* $\delta = \prod_{j=1}^{p} \beta_j$.

1. *If* $\mathbb{Q} \cap [0,1) \subset \mathrm{Per}(\sigma^j(\mathcal{B}))$ *for all* $j \in \mathbb{N}$, *then* δ *is Pisot or Salem and* $\beta_i \in \mathbb{Q}(\delta)$ *for all* $i \in \mathbb{N}$.
2. *If* δ *is Pisot and* $\beta_i \in \mathbb{Q}(\delta)$ *for all* $i \in \mathbb{N}$, *then* $\mathbb{Q}(\delta) \cap [0,1) = \mathrm{Per}(\sigma^j(\mathcal{B}))$ *for all* $j \in \mathbb{N}$.

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2. *If δ is Pisot and $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \mathbb{N}$, then $\mathbb{Q}(\delta) \cap [0,1) = \mathrm{Per}(\sigma^j(\mathcal{B}))$ for all $j \in \mathbb{N}$.*

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Let $\mathcal{B} = (\beta_1, \beta_2, \ldots, \beta_p)$ *be an alternate base,* $\delta = \prod_{j=1}^{p} \beta_j$.

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2. *If* δ *is Pisot and* $\beta_i \in \mathbb{Q}(\delta)$ *for all* $i \in \mathbb{N}$*, then* $\mathbb{Q}(\delta) \cap [0, 1) = \mathrm{Per}(\sigma^j(\mathcal{B}))$ *for all* $j \in \mathbb{N}$.

Sketch of the proof (for base $\mathcal{B} = (\beta_1, \beta_2, \beta_3)$)

$$\frac{1}{s} = \frac{a_1}{\beta_1} + \frac{a_2}{\beta_1\beta_2} + \frac{a_3}{\delta} + \cdots$$

$$\frac{1}{s} = \frac{b_1}{\beta_2} + \frac{b_2}{\beta_2\beta_3} + \frac{b_3}{\delta} + \cdots$$

$$\frac{1}{s} = \frac{c_1}{\beta_3} + \frac{c_2}{\beta_3\beta_1} + \frac{c_3}{\delta} + \cdots$$

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Theorem

For any $\varepsilon > 0$ there exists $A_1 \in \mathbb{N}$ such that for any $A > A_1$, the interval $\big(A, A(1+\varepsilon)\big)$ contains at least one rational prime.

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$$\boldsymbol{\gamma(\mathcal{B})} = \sup\left\{\nu : \forall\, x \in [0, \nu) \cap \mathbb{Q},\ d(\mathcal{B}, x) \text{ purely periodic}\right\} > 0$$

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Class of bases with $\gamma(\mathcal{B}) = 1$:

$\delta > 1$ root of $x^2 - (m+1)x - 1$, $m \geq 1$, $\beta_1 = \frac{\delta}{\delta - 1}$, $\beta_2 = \delta - 1$.

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Theorem (Midy)

Let $q > 5$ be prime. If $\frac{p}{q} \in (0,1)$ has the minimal period of even length then the sum of the first and the second half of the period is a number whose decimal expansion uses only the digit 9.

Midy property in non-integer base

E.g. golden ratio $\tau = \frac{1}{2}(1+\sqrt{5})$:

$$(\tfrac{1}{3})_\tau = 0.(00101000)^\omega, \qquad (\tfrac{1}{5})_\tau = 0.(00010010101001001000)^\omega$$

$$\begin{array}{c} 0010 \\ 1000 \\ \hline 1010 \\ = \tau^4 - 1 \end{array} \qquad \begin{array}{c} 0001001010 \\ 1001001000 \\ \hline 1010101010 \\ = \tau^{10} - 1 \end{array}$$

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$q \in \mathbb{N}$ has the **Midy property in base β** if

$\exists p \in \mathbb{N}$, $p < q$ coprime with q, such that

- $\left(\frac{p}{q}\right)_\beta = 0.(c_1 c_2 \cdots c_{2n})^\omega$ where $2n$ is the shortest period; and
- $x + y = \beta^n - 1$ where $(x)_\beta = c_1 c_2 \cdots c_n$ and $(y)_\beta = c_{n+1} c_{n+2} \cdots c_{2n}$.

Midy property in non-integer base

Let β have the minimal polynomial

$$f(X) = X^d - c_{d-1}X^{d-1} - c_{d-2}X^{d-2} - \cdots - c_1X - c_0 \in \mathbb{Z}[X].$$

The companion matrix of β is

$$C = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 1 & 0 & \cdots & 0 & c_1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & c_{d-2} \\ 0 & 0 & \cdots & 1 & c_{d-1} \end{pmatrix}.$$

C has eigenvalue β to the eigenvector $(1, \beta, \ldots, \beta^{d-1})^\top$.

Midy property in non-integer base

Necessary condition:

Theorem

Let $C \in \mathbb{Z}^{d \times d}$ be the companion matrix of an algebraic integer $\beta > 1$ of degree d. If $q \in \mathbb{N}$, $q > 2$, has the Midy property in base β, then there exists $N \in \mathbb{N}$, $N > 1$ such that $C^N \equiv -I \mod q$.

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Sufficient condition for the golden ratio:

Theorem

Let $C = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, $q \in \mathbb{N}$, $q > 2$. If there exists $N \in \mathbb{N}$, $N > 1$ such that $C^N \equiv -I \mod q$, then q has the Midy property in base τ.

Midy property in base τ and Fibonacci numbers

For the golden ratio $C^N = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)^N = \begin{pmatrix} F_{N-1} & F_N \\ F_N & F_{N+1} \end{pmatrix}$.

Condition $C^N \equiv -I \mod q$ is equivalent to $q | F_N, \quad q | F_{N\pm1} + 1$.

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Let $q \in \mathbb{N}, q > 2$.

1. *If q is a divisor of F_{2n-1} for some $n \in \mathbb{N}, n \geq 3$, then q has the Midy property in base τ.*
2. *If q is a multiple of F_{2n} for some $n \in \mathbb{N}, n \geq 3$, then q does not have the Midy property in base τ.*

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Sloane's On-Line Encyclopedia of Integer Sequences A001177:

$a(m)$ the smallest in $\mathbb{N}$ such that m divides $F_{a(m)}$.

Results for primes

Theorem

Let $q > 2$ *be a prime,* $q = 5$ *or* $q \equiv \pm 2 \mod 5$*. Then* q *has the Midy property in base* τ*.*

Theorem

Let $q \in \mathbb{N}$ *be a prime such that* $q \equiv 11 \mod 20$ *or* $q \equiv 19 \mod 20$*. Then* q *does not have the Midy property in base* τ*.*

Midy property in other bases

$\delta = \frac{1}{2}(3 + \sqrt{13})$, root of $x^2 - 3x - 1$, $d_\delta^*(1) = (30)^\omega$

$d(\delta, \frac{1}{5}) = (020130220110)^\omega$

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$\mathcal{B} = \left(\frac{\delta}{\delta - 1}, \delta - 1\right)$, $d^*(\mathcal{B}, 1) = (10)^\omega$

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$$\begin{array}{c} 000200010300 \\ 020200010100 \\ \hline 020400020400 \\ \hline 101010101010 \end{array}$$

Thank you!