Expansions of generalized Thue-Morse numbers

Yao-Qiang Li
School of Mathematics and Statistics
Guangdong University of Technology, China
yaoqiang.li@gdut.edu.cn; scutyaoqiangli@qq.com

Background

Expansions in non-integer bases are pioneered by Rényi and Parry in 1957-1960.

Given $m \in \mathbb{N}$, a base $\beta \in (1, m+1]$ and $x \in \mathbb{R}$, a sequence $w = (w_n)_{n \geq 1} \in \{0, 1, \ldots, m\}^\infty$ is called a $\beta$-expansion of $x$ if

$$x = \pi_\beta(w) := \sum_{n=1}^{\infty} \frac{w_n}{\beta^n}$$

• $x$ has a $\beta$-expansion iff $x \in [0, \frac{1}{\beta-1}]$.
• $x$ may have many $\beta$-expansions or have a unique one.

Unique expansions have attracted a lot of attention in the last four decades: Allouche, Baker, Clarke, de Vries, Erdős, Frougny, Glingenstein, Horváth, Joó, Kalle, Komornik, Kong, Li, Loret, Lü, Sidorov, ...

The famous Thue-Morse sequence $(t_n)_{n \geq 0}$ is

$$t_0 = 0, \quad t_1 = 1, \quad t_{n+2} = t_n t_{n+1}.$$  

The shifted Thue-Morse sequence $(s_n)_{n \geq 1}$ is

$$s_0 = 1, \quad s_1 = 0, \quad s_{n+2} = s_n s_{n+1}.$$  

**Main Results**

**Proposition.** Let $\sigma$ be the shift map. The following are all equivalent.

1. For all $0 < n < q - 1$ we have $\theta_1 \cdots \theta_{q-n} \neq 0$.
2. For all $n \geq 1$ we have $\sigma^n \theta < \theta$.
3. For all $n > 1$ we have $\sigma^n \theta < \theta$.
4. For all $n \geq 1$ we have $\sigma^n \theta > \theta$.
5. Whenever $\sigma^n \theta < \theta$.
6. Whenever $\sigma^n \theta > \theta$.

**Theorem.** (1) For all $\beta \in (1, m+1]$, if $\theta$ is the greedy, lazy, quasi-greedy, quasi-lazy or unique $\beta$-expansion of $\pi_\beta(\theta)$, then $\beta \geq \beta_0$.

(2) The following are all equivalent.
1. For all $0 < n < q - 1$ we have $\theta_1 \cdots \theta_{q-n} < \theta_1 \cdots \theta_q$.
2. $\theta$ is the unique $\beta_0$-expansion of 1.
3. $\theta$ is the greedy $\beta_0$-expansion of 1.
4. $\theta$ is the lazy $\beta_0$-expansion of 1.
5. $\theta$ is the quasi-greedy $\beta_0$-expansion of 1.
6. $\theta$ is the quasi-lazy $\beta_0$-expansion of 1.

**Corollary.** Let $\beta \in (1, 2]$ and consider the alphabet $\{0, 1\}$. The following are all equivalent.

1. $(t_n)_{n \geq 0}$ is the unique $\beta$-expansion of $\pi_\beta((t_n)_{n \geq 0})$.
2. $(t_n)_{n \geq 0}$ is the greedy $\beta$-expansion of $\pi_\beta((t_n)_{n \geq 0})$.
3. $(t_n)_{n \geq 0}$ is the lazy $\beta$-expansion of $\pi_\beta((t_n)_{n \geq 0})$.
4. $(t_n)_{n \geq 0}$ is the quasi-greedy $\beta$-expansion of $\pi_\beta((t_n)_{n \geq 0})$.
5. $(t_n)_{n \geq 0}$ is the quasi-lazy $\beta$-expansion of $\pi_\beta((t_n)_{n \geq 0})$.
6. $\beta \geq \beta_0$, the classical Komornik-Loreti constant.

Questions

When will $\theta$ be the unique $\beta$-expansion of $\pi_\beta(\theta)$?

- **greedy** (maximal among all the $\beta$-expansions of $x$)
- **lazy** (minimal among all the $\beta$-expansions of $x$)
- **quasi-greedy** (maximal among all the $\beta$-expansions of $x$ not ending with $0^\infty$)
- **quasi-lazy** (minimal among all the $\beta$-expansions of $x$ not ending with $m^\infty$)

In 2015, Kong and Li [1, Theorem 4.4] showed that:

$\theta$ is the unique $\beta_0$-expansion of $1^\infty$ if for all $1 \leq n \leq q$ we have

$$\theta_1 \cdots \theta_q \leq \theta_q \cdots \theta_1$$

References: