Expansions of generalized Thue-Morse numbers

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Numeration 2024
3-7 Jun 2024 Utrecht (Netherlands)

Background

Expansions in non-integer bases are pioneered by Rényi and Parry in 1957-1960.

Given $m \in \mathbb{N}$, a base $\beta \in (1, m+1]$ and $x \in \mathbb{R}$, a sequence $w = (w_n)_{n \geqslant 1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ is called a β-expansion of x if

$$x = \pi_\beta(w) := \sum_{n=1}^{\infty} \frac{w_n}{\beta^n}.$$

- x has a β-expansion iff $x \in [0, \frac{m}{\beta-1}]$.
- x may have many β-expansions or have a unique one.

Unique expansions have attracted a lot of attention in the last four decades: Allouche, Baker, Clarke, de Vries, Erdös, Frougny, Glendinning, Horváth, Joó, Kalle, Komornik, Kong, Li, Loreti, Lü, Sidorov, ...

The famous Thue-Morse sequence $(t_n)_{n \geqslant 0}$ is

$$0110\ 1001\ 1001\ 0110\ 1001\ 0110\ 0110\ 1001\ \cdots.$$

The shifted Thue-Morse sequence $(t_n)_{n \geqslant 1}$ is

$$1101\ 0011\ 0010\ 1101\ 0010\ 1100\ 1101\ 0011\ \cdots.$$

Let $\overline{0} := 1$ and $\overline{1} := 0$. Then $(t_n)_{n \geqslant 1}$ can be defined by

$$t_1 := 1,\ t_1t_2 := t_1\overline{t_1}^+ = 11,\ t_1t_2t_3t_4 := t_1t_2\overline{t_1t_2}^+ = 1101,\ \cdots.$$

The classical Komornik-Loreti constant is the unique $\mathfrak{q} \in (1,2)$ s.t. $\sum_{n=1}^{\infty} \frac{t_n}{\mathfrak{q}^n} = 1$.

Generalization: For $k \in \{0, \cdots, m\}$ denote $\overline{k} := m - k$.
For $m, q \in \mathbb{N}$ and $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \neq 0$, define the $(m; \theta_1, \cdots, \theta_q)$-Thue-Morse sequence $\theta = (\theta_n)_{n \geqslant 1}$ by

$$\theta_1 \cdots \theta_{2q} := \theta_1 \cdots \theta_q \overline{\theta_1 \cdots \theta_q}^+,$$
$$\theta_1 \cdots \theta_{4q} := \theta_1 \cdots \theta_{2q} \overline{\theta_1 \cdots \theta_{2q}}^+,$$
$$\cdots.$$

These sequences were previously studied by Allouche, de Vries, Komornik, Kong, Li, $\cdots$ in different terms.

The $(m; \theta_1, \cdots, \theta_q)$-Komornik-Loreti constant is the unique $\beta_\theta \in (1, m+1)$ s.t.

$$\sum_{n=1}^{\infty} \frac{\theta_n}{\beta_\theta^n} = 1.$$

For any $(m; \theta_1, \cdots, \theta_q)$-Thue-Morse sequence $\theta = (\theta_i)_{i \geqslant 1}$ and $\beta \in (1, m+1]$, we call

$$\pi_\beta(\theta) = \sum_{n=1}^{\infty} \frac{\theta_n}{\beta^n}$$

the β-$(m; \theta_1, \cdots, \theta_q)$-Thue-Morse number.

Questions

When will θ be the unique β-expansion of $\pi_\beta(\theta)$?

greedy (maximal among all the β-expansions of x)
lazy (minimal among all the β-expansions of x)
quasi-greedy (maximal among all the β-expansions of x not end with 0^∞)
quasi-lazy (minimal among all the β-expansions of x not end with m^∞)

- In 2015, Kong and Li [1, Theorem 4.4] showed that:

θ is the unique β_θ-expansion of 1 iff for all $1 \leqslant n \leqslant q$ we have

$$\overline{\theta_1 \cdots \theta_q^-} \leqslant \theta_n \cdots \theta_q^- \theta_1 \cdots \theta_{n-1} \quad \text{and} \quad \theta_n \cdots \theta_q \overline{\theta_1 \cdots \theta_{n-1}} \leqslant \theta_1 \cdots \theta_q.$$

Main Results

Proposition. Let σ be the shift map. The following are all equivalent.
(1) For all $0 \leqslant n \leqslant q-1$ we have $\overline{\theta_1 \cdots \theta_{q-n}} < \theta_{n+1} \cdots \theta_q \leqslant \theta_1 \cdots \theta_{q-n}$.
(2) For all $n \geqslant 1$ we have $\overline{\theta} < \sigma^n\theta < \theta$.
(3) For all $n \geqslant 1$ we have $\sigma^n\theta < \theta$.
(4) For all $n \geqslant 1$ we have $\sigma^n\theta > \overline{\theta}$.
(5) Whenever $\theta_n < m$ we have $\sigma^n\theta < \theta$.
(6) Whenever $\theta_n > 0$ we have $\sigma^n\theta > \overline{\theta}$.

Theorem. (1) For all $\beta \in (1, m+1]$, if θ is the greedy, lazy, quasi-greedy, quasi-lazy or unique β-expansion of $\pi_\beta(\theta)$, then $\beta \geqslant \beta_\theta$.
(2) The following are all equivalent.
① For all $0 \leqslant n \leqslant q-1$ we have $\overline{\theta_1 \cdots \theta_{q-n}} < \theta_{n+1} \cdots \theta_q \leqslant \theta_1 \cdots \theta_{q-n}$.
② θ is the unique β_θ-expansion of 1.
③ θ is the greedy β_θ-expansion of 1.
④ θ is the lazy β_θ-expansion of 1.
⑤ θ is the quasi-greedy β_θ-expansion of 1.
⑥ θ is the quasi-lazy β_θ-expansion of 1.
⑦ $\{\beta \in (1, m+1] : \theta$ is the unique β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.
⑧ $\{\beta \in (1, m+1] : \theta$ is the greedy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.
⑨ $\{\beta \in (1, m+1] : \theta$ is the lazy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.
⑩ $\{\beta \in (1, m+1] : \theta$ is the quasi-greedy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.
⑪ $\{\beta \in (1, m+1] : \theta$ is the quasi-lazy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.

Corollary. Let $\beta \in (1, 2]$ and consider the alphabet $\{0, 1\}$.
The following are all equivalent.
(1) $(t_n)_{n \geqslant 1}$ is the unique β-expansion of $\pi_\beta((t_n)_{n \geqslant 1})$.
(2) $(t_n)_{n \geqslant 1}$ is the greedy β-expansion of $\pi_\beta((t_n)_{n \geqslant 1})$.
(3) $(t_n)_{n \geqslant 1}$ is the lazy β-expansion of $\pi_\beta((t_n)_{n \geqslant 1})$.
(4) $(t_n)_{n \geqslant 1}$ is the quasi-greedy β-expansion of $\pi_\beta((t_n)_{n \geqslant 1})$.
(5) $(t_n)_{n \geqslant 1}$ is the quasi-lazy β-expansion of $\pi_\beta((t_n)_{n \geqslant 1})$.
(6) $\beta \geqslant$ the classical Komornik-Loreti constant.

References: [1] D. Kong, W. Li, Hausdorff dimension of unique beta expansions. Nonlinearity 28 (2015), no. 1, 187-209.
[2] Y.-Q. Li, Expansions of generalized Thue-Morse numbers. Adv. in Appl. Math. 143 (2023), Paper No. 102456.