Expansions of generalized Thue-Morse numbers

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1. Background

2. Main Results
1. Background
Expansions in non-integer bases are pioneered by Rényi and Parry in 1957-1960.

Given $m \in \mathbb{N}$, a base $\beta \in (1, m+1]$ and $x \in \mathbb{R}$, a sequence $w = (w_n)_{n \geq 1} \in \{0, 1, \ldots, m\}^\mathbb{N}$ is called a $\beta$-expansion of $x$ if

$$x = \pi_\beta(w) := \sum_{n=1}^{\infty} \frac{w_n}{\beta^n}.$$ 

- $x$ has a $\beta$-expansion iff $x \in [0, \frac{m}{\beta-1}]$.
- An $x$ may have many $\beta$-expansions or have a unique one.

Unique expansions have attracted a lot of attention in the last four decades: Allouche, Baker, Clarke, de Vries, Erdös, Frougny, Glendinning, Horváth, Joó, Kalle, Komornik, Kong, Li, Loreti, Lü, Sidorov, ...
The famous Thue-Morse sequence \((t_n)_{n \geq 0}\) is
\[
0110 \ 1001 \ 1001 \ 0110 \ 1001 \ 0110 \ 0110 \ 1001 \ \cdots
\]
The shifted Thue-Morse sequence \((t_n)_{n \geq 1}\) is
\[
1101 \ 0011 \ 0010 \ 1101 \ 0010 \ 1100 \ 1101 \ 0011 \ \cdots
\]
Let \(0 := 1\) and \(1 := 0\). Then \((t_n)_{n \geq 1}\) can be defined by
\[
t_1 := 1,
\]
\[
t_1 t_2 := t_1 t_1^{\ominus} = 11,
\]
\[
t_1 t_2 t_3 t_4 := t_1 t_2 t_1 t_2^{\ominus} = 1101,
\]
\[
t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 := t_1 t_2 t_3 t_4 t_1 t_2 t_3 t_4^{\ominus} = 11010011,
\]
\[
\cdots
\]
The classical Komornik-Loreti constant is the unique \(q \in (1, 2)\) s.t.
\[
\sum_{n=1}^{\infty} \frac{t_n}{q^n} = 1.
\]
Generalization: For $k \in \{0, \cdots, m\}$ denote $\overline{k} := m - k$.
For $m, q \in \mathbb{N}$ and $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \neq 0$, define the $(m; \theta_1, \cdots, \theta_q)$-Thue-Morse sequence $\theta = (\theta_n)_{n \geq 1}$ by

$$
\theta_1 \cdots \theta_{2q} := \theta_1 \cdots \theta_q \overline{\theta_1 \cdots \theta_q}, \\
\theta_1 \cdots \theta_{4q} := \theta_1 \cdots \theta_{2q} \overline{\theta_1 \cdots \theta_{2q}}, \\
\theta_1 \cdots \theta_{8q} := \theta_1 \cdots \theta_{4q} \overline{\theta_1 \cdots \theta_{4q}}, \\
 \vdots
$$

These sequences were previously studied by Allouche, de Vries, Komornik, Kong, Li, \cdots in different terms.

The $(m; \theta_1, \cdots, \theta_q)$-Komornik-Loreti constant is the unique $\beta_\theta \in (1, m + 1)$ s.t.

$$
\sum_{n=1}^{\infty} \frac{\theta_n}{\beta_\theta^n} = 1.
$$
For any \((m; \theta_1, \cdots, \theta_q)\)-Thue-Morse sequence \(\theta = (\theta_i)_{i \geq 1}\) and \(\beta \in (1, m+1]\), we call
\[
\pi_{\beta}(\theta) = \sum_{n=1}^{\infty} \frac{\theta_n}{\beta^n}
\]
the \(\beta-(m; \theta_1, \cdots, \theta_q)\)-Thue-Morse number.

**Question:** When will this expansion be unique?

- **greedy** (maximal among all the \(\beta\)-expansions of \(x\))
- **lazy** (minimal among all the \(\beta\)-expansions of \(x\))
- **quasi-greedy** (maximal among all the \(\beta\)-expansions of \(x\) not end with \(0^\infty\))
- **quasi-lazy** (minimal among all the \(\beta\)-expansions of \(x\) not end with \(m^\infty\))
D. Kong and W. Li (2015):

\[ \theta \text{ is the unique } \beta_\theta\text{-expansion of } 1 \]

iff

\[ \theta_1 \cdots \theta_q \leq \theta_n \cdots \theta_q \theta_1 \cdots \theta_{n-1} \quad \text{and} \quad \theta_n \cdots \theta_q \theta_1 \cdots \theta_{n-1} \leq \theta_1 \cdots \theta_q \]

for all \( 1 \leq n \leq q \).
2. Main Results
Let $m, q \in \mathbb{N}$, $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \neq 0$ and $\theta = (\theta_n)_{n \geq 1}$ be the $(m; \theta_1, \cdots, \theta_q)$-Thue-Morse sequence. Use $\sigma$ to denote the shift map.

**Proposition (L., Adv. in Appl. Math., 2023)**

The following are all equivalent.

(1) For all $0 \leq n \leq q - 1$ we have $\theta_1 \cdots \theta_{q-n} < \theta_{n+1} \cdots \theta_q \leq \theta_1 \cdots \theta_{q-n}$.

(2) For all $n \geq 1$ we have $\overline{\theta} < \sigma^n \theta < \theta$.

(3) For all $n \geq 1$ we have $\sigma^n \theta < \theta$.

(4) For all $n \geq 1$ we have $\sigma^n \theta > \overline{\theta}$.

(5) Whenever $\theta_n < m$ we have $\sigma^n \theta < \theta$.

(6) Whenever $\theta_n > 0$ we have $\sigma^n \theta > \overline{\theta}$.
Recall the \((m; \theta_1, \cdots, \theta_q)\)-Komornik-Loreti constant \(\beta_\\theta\).

**Theorem (L., Adv. in Appl. Math., 2023)**

1. For all \(\beta \in (1, m + 1]\), if \(\theta\) is the greedy, lazy, quasi-greedy, quasi-lazy or unique \(\beta\)-expansion of \(\pi_\beta(\theta)\), then \(\beta \geq \beta_\\theta\).
2. The following are all equivalent.
   1. For all \(0 \leq n \leq q - 1\) we have \(\theta_1 \cdots \theta_{q-n} < \theta_{n+1} \cdots \theta_q \leq \theta_1 \cdots \theta_{q-n}\).
   2. \(\theta\) is the unique \(\beta_\\theta\)-expansion of 1.
   3. \(\theta\) is the greedy \(\beta_\\theta\)-expansion of 1.
   4. \(\theta\) is the lazy \(\beta_\\theta\)-expansion of 1.
   5. \(\theta\) is the quasi-greedy \(\beta_\\theta\)-expansion of 1.
   6. \(\theta\) is the quasi-lazy \(\beta_\\theta\)-expansion of 1.
   7. \(\{\beta \in (1, m + 1] : \theta\) is the unique \(\beta\)-expansion of \(\pi_\beta(\theta)\}\} = [\beta_\\theta, m + 1]\).
   8. \(\{\beta \in (1, m + 1] : \theta\) is the greedy \(\beta\)-expansion of \(\pi_\beta(\theta)\}\} = [\beta_\\theta, m + 1]\).
   9. \(\{\beta \in (1, m + 1] : \theta\) is the lazy \(\beta\)-expansion of \(\pi_\beta(\theta)\}\} = [\beta_\\theta, m + 1]\).
   10. \(\{\beta \in (1, m + 1] : \theta\) is the quasi-greedy \(\beta\)-expansion of \(\pi_\beta(\theta)\}\} = [\beta_\\theta, m + 1]\).
   11. \(\{\beta \in (1, m + 1] : \theta\) is the quasi-lazy \(\beta\)-expansion of \(\pi_\beta(\theta)\}\} = [\beta_\\theta, m + 1]\).
Recall the classical Thue-Morse sequence \((t_n)_{n \geq 0}\).

Corollary (L., Adv. in Appl. Math., 2023)

Let \(\beta \in (1, 2]\) and consider the alphabet \(\{0, 1\}\). The following are all equivalent.

1. \((t_n)_{n \geq 1}\) is the unique \(\beta\)-expansion of \(\pi_\beta((t_n)_{n \geq 1})\).
2. \((t_n)_{n \geq 1}\) is the greedy \(\beta\)-expansion of \(\pi_\beta((t_n)_{n \geq 1})\).
3. \((t_n)_{n \geq 1}\) is the lazy \(\beta\)-expansion of \(\pi_\beta((t_n)_{n \geq 1})\).
4. \((t_n)_{n \geq 1}\) is the quasi-greedy \(\beta\)-expansion of \(\pi_\beta((t_n)_{n \geq 1})\).
5. \((t_n)_{n \geq 1}\) is the quasi-lazy \(\beta\)-expansion of \(\pi_\beta((t_n)_{n \geq 1})\).
6. \(\beta \geq\) the classical Komornik-Loreti constant.
Thank you!