Expansions of generalized Thue-Morse numbers

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1. Background

Expansions in non-integer bases are pioneered by Rényi and Parry in 1957-1960.

Given $m \in \mathbb{N}$, a base $\beta \in (1, m+1]$ and $x \in \mathbb{R}$, a sequence $w = (w_n)_{n \geqslant 1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ is called a β-expansion of x if

$$x = \pi_\beta(w) := \sum_{n=1}^{\infty} \frac{w_n}{\beta^n}.$$

- x has a β-expansion iff $x \in [0, \frac{m}{\beta - 1}]$.
- An x may have many β-expansions or have a unique one.

Unique expansions have attracted a lot of attention in the last four decades: Allouche, Baker, Clarke, de Vries, Erdös, Frougny, Glendinning, Horváth, Joó, Kalle, Komornik, Kong, Li, Loreti, Lü, Sidorov, ...

Thue-Morse sequence and Komornik-Loreti constant

The famous Thue-Morse sequence $(t_n)_{n\geqslant 0}$ is

$$0110\ 1001\ 1001\ 0110\ 1001\ 0110\ 0110\ 1001\ \cdots.$$

The shifted Thue-Morse sequence $(t_n)_{n\geqslant 1}$ is

$$1101\ 0011\ 0010\ 1101\ 0010\ 1100\ 1101\ 0011\ \cdots.$$

Let $\overline{0} := 1$ and $\overline{1} := 0$. Then $(t_n)_{n\geqslant 1}$ can be defined by

$$\begin{aligned}
&t_1 := 1,\\
&t_1 t_2 := t_1 \overline{t_1}^{+} = 11,\\
&t_1 t_2 t_3 t_4 := t_1 t_2 \overline{t_1 t_2}^{+} = 1101,\\
&t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 := t_1 t_2 t_3 t_4 \overline{t_1 t_2 t_3 t_4}^{+} = 11010011,\\
&\cdots,
\end{aligned}$$

The classical Komornik-Loreti constant is the unique $\mathfrak{q} \in (1,2)$ s.t.

$$\sum_{n=1}^{\infty} \frac{t_n}{\mathfrak{q}^n} = 1.$$

Generalization: For $k \in \{0, \cdots, m\}$ denote $\overline{k} := m - k$.
For $m, q \in \mathbb{N}$ and $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \neq 0$, define the $(m; \theta_1, \cdots, \theta_q)$-Thue-Morse sequence $\theta = (\theta_n)_{n \geqslant 1}$ by

$$\begin{aligned} \theta_1 \cdots \theta_{2q} &:= \theta_1 \cdots \theta_q \, \overline{\theta_1 \cdots \theta_q}^+, \\ \theta_1 \cdots \theta_{4q} &:= \theta_1 \cdots \theta_{2q} \, \overline{\theta_1 \cdots \theta_{2q}}^+, \\ \theta_1 \cdots \theta_{8q} &:= \theta_1 \cdots \theta_{4q} \, \overline{\theta_1 \cdots \theta_{4q}}^+, \\ &\vdots \end{aligned}$$

These sequences were previously studied by Allouche, de Vries, Komornik, Kong, Li, $\cdots$ in different terms.

The $(m; \theta_1, \cdots, \theta_q)$-Komornik-Loreti constant is the unique $\beta_\theta \in (1, m+1)$ s.t.

$$\sum_{n=1}^{\infty} \frac{\theta_n}{{\beta_\theta}^n} = 1.$$

For any $(m; \theta_1, \cdots, \theta_q)$-Thue-Morse sequence $\theta = (\theta_i)_{i \geqslant 1}$ and $\beta \in (1, m+1]$, we call

$$\pi_\beta(\theta) = \sum_{n=1}^{\infty} \frac{\theta_n}{\beta^n}$$

the β-$(m; \theta_1, \cdots, \theta_q)$-Thue-Morse number.

Question: When will this expansion be unique?

greedy (maximal among all the β-expansions of x)

lazy (minimal among all the β-expansions of x)

quasi-greedy (maximal among all the β-expansions of x not end with 0^∞)

quasi-lazy (minimal among all the β-expansions of x not end with m^∞)

D. Kong and W. Li (2015):

θ is the unique β_θ-expansion of 1

iff

$$\overline{\theta_1 \cdots \theta_q^-} \leqslant \theta_n \cdots \theta_q^- \theta_1 \cdots \theta_{n-1} \quad \text{and} \quad \theta_n \cdots \theta_q \overline{\theta_1 \cdots \theta_{n-1}} \leqslant \theta_1 \cdots \theta_q$$

for all $1 \leqslant n \leqslant q$.

2. Main Results

Let $m, q \in \mathbb{N}$, $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \neq 0$ and $\theta = (\theta_n)_{n \geqslant 1}$ be the $(m; \theta_1, \cdots, \theta_q)$-Thue-Morse sequence. Use σ to denote the shift map.

Proposition (L., Adv. in Appl. Math., 2023)

The following are all equivalent.

(1) *For all* $0 \leqslant n \leqslant q-1$ *we have* $\overline{\theta_1 \cdots \theta_{q-n}} < \theta_{n+1} \cdots \theta_q \leqslant \theta_1 \cdots \theta_{q-n}$.

(2) *For all* $n \geqslant 1$ *we have* $\overline{\theta} < \sigma^n \theta < \theta$.

(3) *For all* $n \geqslant 1$ *we have* $\sigma^n \theta < \theta$.

(4) *For all* $n \geqslant 1$ *we have* $\sigma^n \theta > \overline{\theta}$.

(5) *Whenever* $\theta_n < m$ *we have* $\sigma^n \theta < \theta$.

(6) *Whenever* $\theta_n > 0$ *we have* $\sigma^n \theta > \overline{\theta}$.

Recall the $(m; \theta_1, \cdots, \theta_q)$-Komornik-Loreti constant β_θ.

Theorem (L., Adv. in Appl. Math., 2023)

(1) For all $\beta \in (1, m+1]$, if θ is the greedy, lazy, quasi-greedy, quasi-lazy or unique β-expansion of $\pi_\beta(\theta)$, then $\beta \geqslant \beta_\theta$.
(2) The following are all equivalent.

① For all $0 \leqslant n \leqslant q-1$ we have $\overline{\theta_1 \cdots \theta_{q-n}} < \theta_{n+1} \cdots \theta_q \leqslant \theta_1 \cdots \theta_{q-n}$.

② θ is the unique β_θ-expansion of 1.

③ θ is the greedy β_θ-expansion of 1.

④ θ is the lazy β_θ-expansion of 1.

⑤ θ is the quasi-greedy β_θ-expansion of 1.

⑥ θ is the quasi-lazy β_θ-expansion of 1.

⑦ $\{\beta \in (1, m+1] : \theta$ is the unique β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.

⑧ $\{\beta \in (1, m+1] : \theta$ is the greedy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.

⑨ $\{\beta \in (1, m+1] : \theta$ is the lazy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.

⑩ $\{\beta \in (1, m+1] : \theta$ is the quasi-greedy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.

⑪ $\{\beta \in (1, m+1] : \theta$ is the quasi-lazy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.

Recall the classical Thue-Morse sequence $(t_n)_{n\geqslant 0}$.

Corollary (L., Adv. in Appl. Math., 2023)

Let $\beta \in (1,2]$ and consider the alphabet $\{0,1\}$. The following are all equivalent.

(1) $(t_n)_{n\geqslant 1}$ *is the unique β-expansion of $\pi_\beta((t_n)_{n\geqslant 1})$.*

(2) $(t_n)_{n\geqslant 1}$ *is the greedy β-expansion of $\pi_\beta((t_n)_{n\geqslant 1})$.*

(3) $(t_n)_{n\geqslant 1}$ *is the lazy β-expansion of $\pi_\beta((t_n)_{n\geqslant 1})$.*

(4) $(t_n)_{n\geqslant 1}$ *is the quasi-greedy β-expansion of $\pi_\beta((t_n)_{n\geqslant 1})$.*

(5) $(t_n)_{n\geqslant 1}$ *is the quasi-lazy β-expansion of $\pi_\beta((t_n)_{n\geqslant 1})$.*

(6) $\beta \geqslant$ *the classical Komornik-Loreti constant.*

Thank you!