# Dependencies between digits of $\beta$-expansions

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Numeration 2024 Utrecht

For real numbers $x \in [0,1)$ and $\beta > 1$, the $\beta$- expansion of $x$ is given by

$$x = \sum_{n>0} x_n \beta^{-n},$$

where the digits $x_i$ are integers such that $0 \leq x_i < \beta$. As such, $\beta$-expansions are defined in the same way as integer bases. However, without further restrictions, this definition leaves redundancy for the expansions of some numbers. In order to fix this problem, when redundancy occurs, only the largest expansion for lexicographic order is kept.

## Definitions and notations

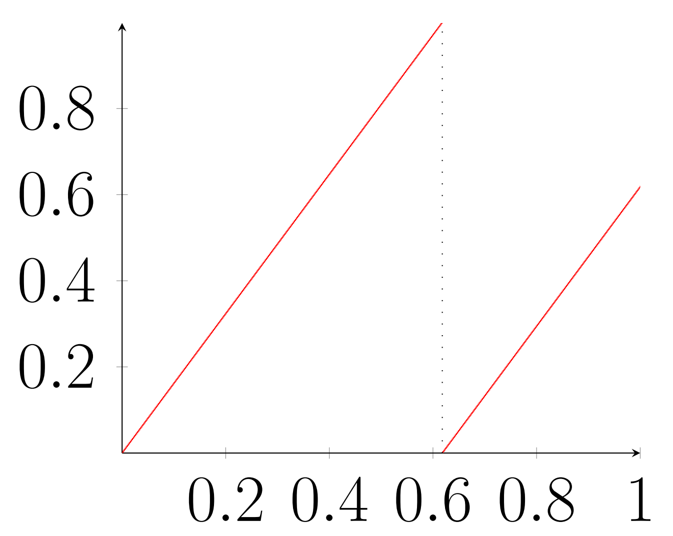
### Expansion of 1

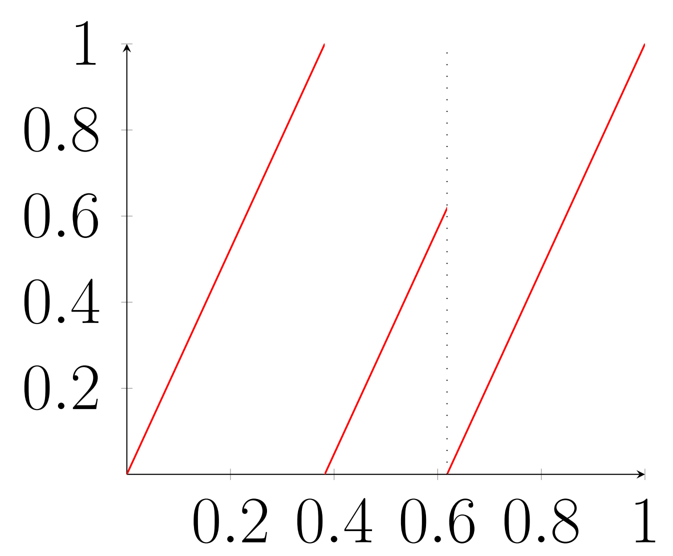
We will denote by $d_\beta(1)$ the expansion of 1 in base $\beta$, and by $(f_n)_{n\geq 0}$ the sequence of the iterated fractional parts of 1, defined as follows :

- $f_0 = 1$,
- For $i \geq 0$, $f_{i+1} = \{\beta f_i\}$, and $d_\beta(1)_i = \lfloor \beta f_i \rfloor$, where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of $x$.

### Cylinders and Parry numbers

- In a given base, a cylinder of order $n$ is a interval made of all the real numbers which expansions in said base begin with the same block of digits of length $n$.
- The restrictions on the digits for $\beta$-expansions have for effect the shortening of some cylinders. The standard cylinders of order $n$ have length $1/\beta^n$, while the short cylinders of order $n$ have length $f_i/\beta^n$ for some $i$.
- The real number $\beta$ is said to be a (resp. simple) Parry number if $d_\beta(1)$ is eventually periodic (resp. finite). It is known for example that Pisot numbers are Parry numbers.
- A real number $\beta$ is a Parry number if, and only if there is finitely many types of cylinders in base $\beta$.




Graphs of $x \mapsto \{\varphi x\}$ and the iterated fonction $x \mapsto \{\varphi\{\varphi x\}\}$ with $\phi = (1 + \sqrt(5))/2$

# Dependencies of digits

As illustrated in the precedent example, consecutive digits of $\beta$-expansions are not independent (here the digit 1 can appear after 0 but not after 1). We will see that even in the Parry case, where there are finitely many cylinder types, the digits are not independents even when far away, using the example of the golden base.

- The measure we used is the Parry measure defined by the density $d\mu_\beta(x) = 1/\Gamma \sum_{n\geq(0)} \mathbb{1}_{x<f_n}\beta^{-n} d\lambda(x)$, where $\Gamma$ is a normalizing constant. It is invariant by the transformation $T_\beta : x \mapsto \{\beta x\}$.

## The example of the golden base

The golden base has the following properties.

- There are two types of cylinders : standard cylinders associated with the digit 0 and cylinders of relative length $\varphi - 1 = 1/\varphi$ associated with the digit 1.
- The zone of weight $(1+1/\varphi)/\Gamma$ contains exactly the expansions that start with 0, and the other weight is $1/\Gamma$.
- As each standard cylinder yields one standard and one short cylinder at the next iteration, and a short cylinder yields one standard cylinder at the next iteration, we can show by induction that at order $n$, there are $F_n$ standard cylinders and $F_{n-1}$ short cylinders, for $F_{n+1}$ cylinders in total, where $F_n$ is Fibonacci's sequence initialized with $F_1 = F_0 = 1$. There are also $F_{n-2}$ cylinders of order $n$ beginning with 0 and ending with 1, as well as $F_{n-3}$ cylinders of order $n$ beginning and ending with 1.

This allows use to compute the following measures :

$$\mu_\varphi(0(\cdot)^n 1) = \frac{(1+1/\varphi)}{\Gamma\varphi^{n+1}} F_n \quad \text{and} \quad \mu_\varphi(1(\cdot)^n 1) = \frac{1}{\Gamma\varphi^{n+1}} F_{n-1}.$$

As Fibonacci's numbers are integers and $1 + 1/\varphi$ is irrational, these two measures cannot be equal.

## Markov's chains and $\beta$-expansions

With the lack of independence between digits, Markov's chain instead. However, in order to give a $\beta$-expansion a structure of Markov's chain, we need to split the digits into different states regrouping smaller sets of cylinders.

- For Lebesgue measure, we only need to consider the states $a_i$ where $a$ is a digit and $i$ is zero for standard cylinders, and the corresponding index in the sequence $f_n$ for short cylinders.
- For Parry measure, we need to take the different weight zones into consideration. Thus we will consider the states $a_{i,j,k}$ with $a$ and $i$ as before, $j$ representing the weight zone that contains the leftmost point of the cylinder, and $k$ representing the type of frontier crossing between zones.

In both cases, we can give Parry numbers expansions a structure of finite state Markov's chains. Indeed, the relative distance of the frontiers to the start of the cylinders that cross them is always an element of the eventually periodic sequence $f_n$. This means there are finitely many cylinders that can cross frontier in finitely many fashion.

For example, let us consider $\beta$ the largest root of $X^3 - X^2 - 2X + 1$. We have $d_\beta(1) = 1(10)^\omega$, and states are $0_{i,j,0}$ and $1_{k,j,0}$ with $i = 0, 3$, $j = 1, 2, 3$ and $k = 1, 2$, as well as $0_{0,1,1}$, $0_{0,2,2}$, $1_{1,1,2}$ and $1_{1,2,1}$.