Dependencies between the digits of β-expansions

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- For $\beta > 1$ and $x \in (0, 1)$, the β-expansion of x is given by $x = \sum_{n>0} x_n \beta^{-n}$ with $0 \leq x_i < \beta$.
- Redundancy is avoided by keeping only the smallest expansion with regards to the lexicographical order for each given x. This leads to the existence of short cylinders, the relative lengths of which follow the sequence $f_\beta(1)$ iterated fractional parts of the expansion of 1.
- Numbers β for which there is finitely many types of cylinders are called *Parry numbers*. For example, the golden number φ is a Parry number, the cylinders of the associated base have length $1/\varphi^n$ or $(\varphi - 1)/\varphi^n$.

- The Parry measure defined by the density $d\mu_\beta(x) = 1/\Gamma \sum_{n\geq(0)} \mathbb{1}_{x<f_\beta(1)_n} d\lambda(x)$ is invariant by the transformation $T_\beta : x \mapsto \{\beta x\}$.
- The digits of a β-expansion for β a non integer are not independent in general whatever the gaps between them, even for Parry numbers.

In base φ, standard intervals each generate one standard interval and one short interval at the next iteration, and short intervals generate one standard interval at the next iteration. Using Fibonacci sequence F_n, we obtain the following.

$$\mu_\varphi(0(\cdot)^n 1) = \frac{(1+1/\varphi)(\varphi-1)}{\Gamma} \frac{F_{n-1}}{\varphi^n} \quad \text{and} \quad \mu_\varphi(1(\cdot)^n 1) = \frac{\varphi-1}{\Gamma} \frac{F_{n=2}}{\varphi^n}.$$

We can use Markov chains to work on the digits of β-expansions, but we need to separate each digit between several state. For Lebesgue measure, we need to separate d between states d_i corresponding to each type of cylinders associated with d. For Parry measure we need to further split the states with regard to the wight zones.

Theorem

When β is a Parry number, the state space of the associated Markov chain is finite.

For example, take β the largest root of $X^3 - X^2 - 2X + 1$. We have $d_\beta(1) = 1(10)^\omega$. The states for Lebesgue measure are 0_0, 1_1, 1_2 ,0_3. The transition probabilities are the following.

- 0_0 is followed by 0_0 with probability $1/\beta$ and by 1_1 with probability $1 - 1/\beta$.
- 1_1 is followed by 0_0 with probability $1/\beta(\beta - 1)$ and by 1_2 with probability $1 - 1/\beta(\beta - 1)$.
- 1_2 is followed by 0_3 with probability 1.
- 0_3 is followed by 0_0 with probability $1/\beta(\beta - 1)$ and by 1^2 with probability $1 - 1/\beta(\beta - 1)$.