Quadratic irrationals and their N-continued fraction expansions

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04-06-2024

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1. Classical, RCF

2. NCF, in particular (N, α)-CF

Act 1

Regular continued fraction expansions

Let $T : [0,1] \to [0,1]$ be defined as
$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

We define $d_1(x) = \lfloor \frac{1}{x} \rfloor$ and $d_n(x) = d_1(T^{n-1}(x))$ for $n \geq 2$.

For $x \in (0,1)$ we find

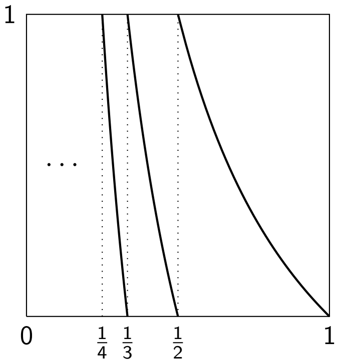

Figure: The map T.

$$x = \frac{1}{d_1(x) + T(x)} = \frac{1}{d_1(x) + \frac{1}{d_2(x) + T^2(x)}} = \frac{1}{d_1(x) + \frac{1}{d_2(x) + \ddots}}$$

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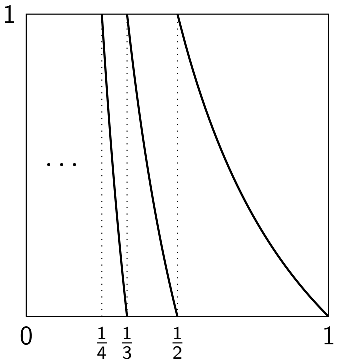

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Quadratic irrationals

Let $x_0 \in [0, 1)$ be a quadratic irrational, i.e. an irrational solution to

$$A_0 x_0^2 + B_0 x_0 + C_0 = 0 \qquad (1)$$

with $A_0, B_0, C_0 \in \mathbb{Z}$.
Then the regular continued fraction expansion of x_0 is periodic.
(The reverse statement also holds.)

Sketch of the proof

The Gauss map, T maps quadratic irrationals to quadratic irrationals.

$$x_1 = T(x_0) = \frac{1}{x_0} - d_1(x_0) \to x_0 = \frac{1}{d_1(x_0) + x_1}$$

We can find A_1, B_1, C_1 such that $A_1 x_1^2 + B_1 x_1 + C_1 = 0$ by substituting x_0 with $\frac{1}{d_1(x_0)+x_1}$ in $A_0 x_0^2 + B_0 x_0 + C_0 = 0$:

$$A_0 \left(\frac{1}{x_1 + d_1}\right)^2 + B_0 \left(\frac{1}{x_1 + d_1}\right) + C_0 = 0 \qquad (2)$$

$$C_0 x_1^2 + (B_0 + 2d_1 C_0) x_1 + A_0 + B_0 d_1 + C_0 d_1^2 = 0, \qquad (3)$$

Sketch of the proof continued

Let $x_n = T^n(x)$. The coefficients $A_n, B_n, C_n \in \mathbb{Z}$ such that $A_n x_n^2 + B_n x + C_n = 0$ can be found recursively:

$$A_{n+1} = C_n \tag{4}$$

$$B_{n+1} = B_n + 2d_{n+1}C_n \tag{5}$$

$$C_{n+1} = A_n + B_n d_{n+1} + C_n d_{n+1}^2 \tag{6}$$

For the determinant one can calculate that the following holds:

$$B_n^2 - 4A_nC_n = B_0^2 - 4A_0C_0. \tag{7}$$

And we can find the bound

$$|A_n| < 2|A_0x| + |A_0| + |B_0|.$$

In particular A_n is unif. bounded. From (4) we find C_n unif. bounded. From (7) we find B_n is unif. bounded.

Act 2

N-continued fractions

Let $N \in \mathbb{N}_{>1}$. We can write any $x \in [0, N]$ as

$$x = \cfrac{N}{d_1(x) + \cfrac{N}{d_2(x) + \ddots}}.$$

In fact, we can do this in many different ways.

$$y = \frac{N}{x} - d \to x = \frac{N}{d + y}$$

As long as $d \geq 1$ and $y \in [0, N]$ we can continue.

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Graph of the branches for $N = 5$

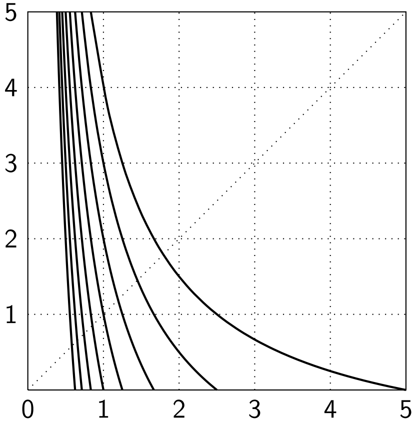

Figure: The branches $\frac{5}{x} - d$.

Making choices

You can choose the branches randomly or follow a certain rule.

- Choose the highest digit possible (greedy).
- Choose the lowest digit possible (lazy).
- Pick the digits that are $1 \mod N$.
- Pick an interval of length 1 such that $y = \frac{N}{x} - d$ is always in the interval.

Different choices, different properties.

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(N, α)-continued fractions

Let $N \in \mathbb{N}_{>1}$ and $\alpha \in (0, \sqrt{N} - 1]$ and define $T_{N,\alpha} : [\alpha, \alpha + 1] \to [\alpha, \alpha + 1)$ as

$$T_{N,\alpha}(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} - \alpha \right\rfloor$$

$$x = \cfrac{N}{d_1(x) + \cfrac{N}{d_2(x) + \ddots}}$$

Introduced and studied by Kraaikamp, L. in 2016. Later studied by Chen, de Jonge, Kraaikamp, Nakada, L., ...

Graph of the maps

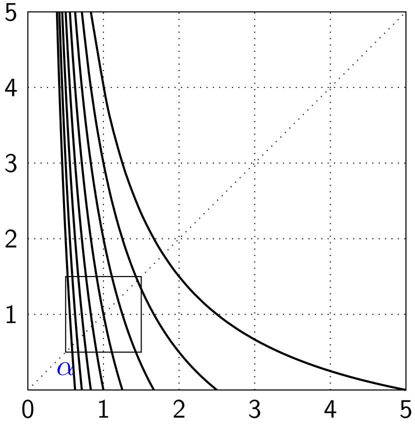


Figure: The branches $\frac{5}{x} - d$.

Periodic points

Periodic points are sometimes rational, sometimes not.
Examples:

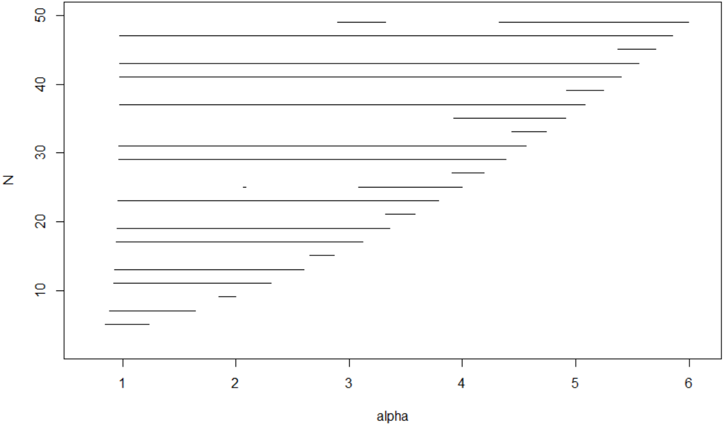
- For $\alpha \in (0,1) \cap (0, \sqrt{N}-1]$ we have $1 = [0; \overline{N-1}]_N$ as solution to $x = \frac{N}{N-1+x}$.
- For $N = 2$ and $\alpha \in (0, \sqrt{2}-1]$ we have $\frac{-3+\sqrt{17}}{2} = [0; \overline{3}]_2$.
- For $N = 3$ and $\alpha = 0.73$ the number $\frac{40}{33}$ is eventually periodic with a pre-period of length 63 and period length 38.

Rational numbers are sometimes periodic, sometimes not.

- For any N and α sufficiently small, all rationals are eventually periodic with tail $[0; \overline{1}]_N$.
- For $N = 3$, all pre-images of 1 are of course periodic. Unknown whether there are a-periodic rational numbers.
- For $N = 7$, and $\alpha \in (1, \sqrt{7}-1)$ all rational numbers are a-periodic (and therefore, all periodic points are quadratic irrationals).

Making the digit set and N co-prime

$$K = \left\{(N,\alpha) : N \in \mathbb{N}_{\geq 2},\ \alpha \in (0, \sqrt{N}-1] \text{ s.t. } \gcd\{N,d\} = 1,\ \forall d \in \mathcal{D}_{N,\alpha}\right\}$$



Non-periodic quadratic irrationals

Theorem (Kraaikamp, L.)

For all $(N, \alpha) \in K$ we have that if $x \in [\alpha, \alpha + 1)$ is an irrational solution to $Ax^2 + Bx + C = 0$ with $A, B, C \in \mathbb{Z}$ co-prime and $\gcd(N, C) = 1$ then x has an a-periodic (N, α)-expansion.

The recurrence relations

Let $x_0 \in [\alpha, \alpha + 1)$ be an irrational solution of

$$A_0 x_0^2 + B_0 x_0 + C_0 = 0 \tag{8}$$

with $A_0, B_0, C_0 \in \mathbb{Z}$ and define $x_n = T_\alpha^n(x)$. Then the coefficients for x_{n+1} are given by

$$A_{n+1} = C_n \tag{9}$$

$$B_{n+1} = NB_n + 2d_{n+1}C_n \tag{10}$$

$$C_{n+1} = N^2 A_n + NB_n d_{n+1} + C_n d_{n+1}^2 \tag{11}$$

For the determinant we find

$$B_n^2 - 4A_n C_n = N^{2n}(B_0^2 - 4A_0 C_0). \tag{12}$$

A_n, B_n, C_n are always co-prime

Fix $(N, \alpha) \in K$. For p prime, with $\gcd(p, N) = 1$: Suppose the contrary and let n be the first time that p divides A_{n+1}, B_{n+1} and C_{n+1}. Then

$$\begin{aligned}
A_{n+1} &= p\widehat{A}_{n+1} = C_n && \text{gives} \quad C_n \equiv 0 \pmod{p}, \\
B_{n+1} &= p\widehat{B}_{n+1} = NB_n + 2d_nC_n && \text{gives} \quad B_n \equiv 0 \pmod{p}, \\
C_{n+1} &= p\widehat{C}_{n+1} = N^2A_n + NB_nd_n + C_nd_n^2 && \text{gives} \quad A_n \equiv 0 \pmod{p}.
\end{aligned}$$

$\rightarrow$ contradiction with minimality of n. For p prime, with $\gcd(p, N) > 1$:

$$C_n \not\equiv 0 \pmod{p} \text{ gives } \begin{cases} A_{n+1} \not\equiv 0 \pmod{p}, \\ B_{n+1} \not\equiv 0 \pmod{p}, \\ C_{n+1} \not\equiv 0 \pmod{p}. \end{cases}$$

In this case we find that p does not divide A_{n+1}, B_{n+1} and C_{n+1} by induction.

Some remarks

1. We can also prove that there are no matching intervals within K.
2. The proof works as long as all digits and C are co-prime with N.
3. For every N there are also quadratic irrationals with infinitely many *different* periodic N-continued fraction expansions. (In order to prove this you can use matching.)
4. For the greedy (and lazy) N-continued fraction it is an open problem whether there exist a-periodic quadratic irrationals.


THANK