Confluent alternate numeration systems

β-numeration systems

A way to represent any nonnegative real number according to a base $\beta > 1$. If $x \in [0,1[$, set $r_0 = x$ then recursively

$$a_{i-1} = \lfloor \beta r_i \rfloor \text{ and } r_{i-1} = \beta r_i - \lfloor \beta r_i \rfloor .$$

The representation is then $0.a_{-1}a_{-2}\cdots$ If $x \geq 1$, divide x by β^k such that $x/\beta^k \in [0,1[$, represent it as above, then put a fractional point after k digits instead of before the first digit.
The word $a_n \cdots a_0.a_{-1}\cdots$ has the value

$$\sum_{j=0}^{n} a_j \beta^j + \sum_{j=1}^{\infty} a_{-j}\beta^{-j}.$$

Examples

– If $\beta = 10$: usual decimal system. $1/7$ is represented by $0.(142857)^\omega$.
–If $\beta = \varphi$, the golden ratio: $1/2$ is represented by $0.(010)^\omega$, 2 is represented by 10.010^ω.

Cantor real numeration systems

Use a periodic sequence of bases instead of one base: $\mathcal{B} = (\beta_i)_{i\in\mathbb{Z}}$, $\beta_{i+p} = \beta_i \forall i$.
We note $S(\mathcal{B}) = (\beta_{i+1})_{i\in\mathbb{Z}}$ and we have $S^p(\mathcal{B}) = \mathcal{B}$.
We can use $(\beta_{-1}, \beta_{-2}, \ldots)$ to represent a number x between 0 and 1: set $r_0 = x$, then recursively

$$a_{i-1} = \lfloor \beta_{i-1} r_i \rfloor \text{ and } r_{i-1} = \beta_{i-1} r_i - \lfloor \beta_{i-1} r_i \rfloor .$$

To represent a number x larger than 1, first divide it by $\beta_0 \cdots \beta_k$ with k minimal such that $x < \beta_0 \cdots \beta_k$, then represent the quotient in base $S^{k+1}(\mathcal{B})$, and finally shift the representation back $k+1$ places.
The word $a_n \cdots a_0.a_{-1}\cdots$ is assigned the value

$$\sum_{j=0}^{n} (a_j \prod_{i=0}^{j-1} \beta_i) + \sum_{j=1}^{\infty} (a_{-j} \prod_{i=-1}^{-j} \frac{1}{\beta_i}).$$

The representation of 1 in the base $(\beta_{i-1}, \beta_{i-2}, \ldots)$ is noted $d_{S^i(\mathcal{B})}(1) = d_{i,1}d_{i,2}\cdots$ and its length is noted m_i.

Examples

Let us consider the example **(a)** on the lower right. We compute $d_{\mathcal{B}}(1)$ and $d_{S(\mathcal{B})}(1)$ in the two tables below

i	$\beta_{i-1} r_i$	a_{i-1}	r_{i-1}
0	$1 \cdot \frac{5+\sqrt{13}}{6} \simeq 1.43$	1	$\frac{-1+\sqrt{13}}{6}$
1	$\frac{-1+\sqrt{13}}{6}\frac{1+\sqrt{13}}{2} = 1$	1	0

i	$\beta_{(i+1)-1} r_i$	a_{i-1}	r_{i-1}
0	$1 \cdot \frac{1+\sqrt{13}}{2} \simeq 2.30$	2	$\frac{-3+\sqrt{13}}{2}$
1	$\frac{-3+\sqrt{13}}{2}\frac{5+\sqrt{13}}{6} \simeq 0.43$	0	$\frac{-1+\sqrt{13}}{6}$
2	$\frac{-1+\sqrt{13}}{6}\frac{1+\sqrt{13}}{2} = 1$	1	0

Thus we have $d_{\mathcal{B}}(1) = 110^\omega$ and $d_{S(\mathcal{B})}(1) = 201^\omega$.
If we consider the example **(b)**, we note that $d_{\mathcal{B}}(1) = 3121$, $d_{S(\mathcal{B})}(1) = 231$ and $d_{S^2(\mathcal{B})}(1) = 123$, omitting trailing zeroes. Note that $d_{i,1} = \lfloor \beta_{i-1} \rfloor$.

Normalisation and Parry conditions

Different words may have the same value. Among these, the one obtained by the greedy algorithm above is called the expansion and is the greatest in radix order. It can be distinguished from the others with this proposition.

An idea of Parry

A finite word $a_n a_{n-1}\cdots a_0.a_{-1}\cdots a_{-m}$ is the expansion of a number x if, and only if, for every position $i \in \{n, \ldots, -m\}$, the word $a_i \cdots a_{-m}$ is lexicographically less than $d_{S^{i+1}(\mathcal{B})}(1)$.

Note that this proposition implies $a_i \leq d_{i+1,1} = \lfloor \beta_i \rfloor$

Examples

In the system **(b)**, the words $1|000|001.$ and $312|101.$ represent the same number (the character | is used to delimit blocks of length p).
Similarly, the words $1|000|210$ and $320|000$ have the same value. Since $320|000 > 3121$, this word cannot be the expansion of its value. On the other hand, since $1|000|210 < 231$, $|210| < 3121$ and $10| < 123$, this word is the expansion of its value.

References

[1] C.Frougny. Confluent linear numeration systems. *Theoretical Computer Science*, 106(2):163–219, 1992
[2] Z.Masáková, E.Pelantová, and K.Studeničová. Rewriting rules for arithmetics in alternate base systems. In F.Drewes and M.Volkov, editors, *Developments in Language Theory*, LNCS 13911, 195–207.

Question

Is it possible to obtain the expansion of a number from any representation, by iteratively rewriting factors that are forbidden by Parry conditions?

Rewriting rules and confluence

From our numeration system, we define a rewriting system. The alphabet is $A = \{0, \ldots, \lfloor \beta_{p-1} \rfloor\} \times \cdots \times \{0, \ldots, \lfloor \beta_0 \rfloor\}$. We allow rewritings of the form

$$\begin{aligned} &\cdots |a_{p-1}\cdots a_{i+1} \quad a_i \quad a_{i-1} \quad a_{i-2}\cdots \\ \to &\cdots |a_{p-1}\cdots a_{i+1}\ (a_i+1)\ (a_{i-1}-d_{i,1})\ (a_{i-2}-d_{i,2})\cdots \end{aligned}$$

as long as the resulting word is still in the allowed alphabet. We also allow $u\alpha v \to u\beta v$ if $\alpha \to \beta$ conforms to the specification above. The reflexive transitive closure of $\to$ is noted $\to^*$. We say that this system is *confluent* if

$$\forall\, u, x, y,\ (u \to^* x \wedge u \to^* y) \Rightarrow \exists v : (x \to^* v \wedge y \to^* v)$$

and we wish to find conditions on our numeration system for this to be the case.

Conditions for confluence

We sketch our main result through two examples. First, consider the case of the system **(a)**. We have $d_{\mathcal{B}}(1) = 11$ and $d_{S(\mathcal{B})}(1) = 201$. Here, the alphabet is $\{0,1\} \times \{0,1,2\}$ and the core rules are $0|11 \to 1|00$ and $02|01 \to 10|00$. We note that the digit 0 in $d_{S(\mathcal{B})}(1)$ is smaller than the maximal allowed digit in this position, which is 1. We will create words u, x, y that disprove confluence. We construct them such that $u \to x$ uses the core rule starting from 0201 and $u \to y$ uses the rule increasing this digit 0.

u_1	0 2	0 1	
u_2		0 2	0 1
$u = \max(u_1, u_2)$	0 2	0 2	0 1
x	1 0	0 1	0 1
y	0 2	1 0	0 0

Now $u \to x$, $u \to y$, but there is no v with $x \to^* v$, $y \to^* v$. This construction can be mimicked as long as one of the non-final digits in a $d_{S^i(\mathcal{B})}(1)$ is smaller than the maximal allowed digit.
Now, consider the system **(b)**. The alphabet is $\{0,1,2,3\} \times \{0,1\} \times \{0,1,2\}$ and the core rules are $0|312|1 \to 1|000|0$, $02|31 \to 10|00$ and $012|3 \to 100|0$. To decide confluence, we need only look at pairs of rules which overlap. In our example, the first digit at the start of a rule cannot be the maximal allowed digit, while all but the first and last digits must be maximal. This implies that applications of rules can overlap by at most one position, which in turn implies confluence.

Proposition

The rewriting system on the alphabet A associated with the numeration system $\mathcal{B}$ is confluent if and only if $d_{i,j} = \lfloor \beta_{i-j} \rfloor$ for all i and all j less than m_i.

Spectrum and $\mathcal{B}$-integers

The *spectrum* of $\mathcal{B}$ is the set of values of words on A that have no fractional part, while the *$\mathcal{B}$-integers* are the numbers whose expansion has no fractional part. In the system **(b)**, those two sets are equal, while in the system **(a)**, we have that $2\beta_0 + 1$ is in the spectrum as it is the value of $210.$, but it is not a β-integer as its expansion is 1001.01.

Proposition

The spectrum of $S^i(\mathcal{B})$ and the set of $S^i(\mathcal{B})$-integers are equal for all i if, and only if $d_{i,j} = \lfloor \beta_{i-j} \rfloor$ for all i and all j less than m_i.

Sketch of proof: to show that the two sets are not equal, use the words x and y from the confluence proof above. To show that the two sets are equal, use rewriting rules to transform a word representing a number into the expansion of that number without ever adding digits to the right of this word.

Numeration systems used as examples

(a) Take $\mathcal{B}$ the 2-periodic sequence defined by $\beta_1 = \dfrac{5+\sqrt{13}}{6}$ and $\beta_0 = \dfrac{1+\sqrt{13}}{2}$.
(b) Let δ be the positive root of $x^3 - 23x^2 + 19x - 4$ (a Pisot number), then take $\mathcal{B}$ the 3-periodic sequence defined by

$$\beta_2 = \frac{\delta^2 - 7\delta + 3}{4\delta - 2},\ \beta_1 = \frac{4\delta - 2}{2\delta} \text{ and } \beta_0 = \frac{2\delta^2}{\delta^2 - 7\delta + 3}.$$

We have $(\beta_2, \beta_1, \beta_0) \simeq (3.909, 1.954, 2.898)$.
(c) Let $(\beta_5, \ldots, \beta_0) = (122/23, 230/33, 99/16, 4, 4, 4)$. The representations $d_{\mathcal{B}}(1)$ through $d_{S^5(\mathcal{B})}(1)$ are $5203, 4, 4, 4, 603, 66$. The associated system is not confluent on the alphabet A, but it is if the digit 4 is excluded from the last three positions.

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