Confluent alternate numeration systems

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β-numeration systems

Represent numbers by words using a *base* β.
The word $a_n \cdots a_0.a_{-1}a_{-2}\cdots$ has value

$$\sum_{j=0}^{n} a_j \beta^j + \sum_{j=1}^{\infty} a_{-j}\beta^{-j}$$

$$= \sum_{j=0}^{n}(a_j \prod_{i=0}^{j-1} \beta) + \sum_{j=1}^{\infty}(a_{-j} \prod_{i=-1}^{-j} \beta^{-1}).$$

Cantor real bases

Generalise from the base sequence $\ldots, \beta, \beta, \ldots$ to the sequence $\ldots \beta_1, \beta_0.\beta_{-1}, \ldots$.
The word $a_n \cdots a_0.a_{-1}a_{-2} \cdots$ has value

$$\sum_{j=0}^{n}(a_j \prod_{i=0}^{j-1} \beta_i) + \sum_{j=1}^{\infty}(a_{-j} \prod_{i=-1}^{-j} \beta_i^{-1}).$$

Alternate numeration system: the base sequence is periodic.

Ambiguity

Different words may have the same value.

Base $(\ldots, 10, 10, 10, \ldots)$: the words 1.0^ω and 0.9^ω both have value 1.

Base $(\ldots, \beta_1, \beta_0.\beta_1, \beta_0, \beta_1, \ldots)$ with $\beta_1 = \dfrac{5+\sqrt{13}}{6}$ and $\beta_0 = \dfrac{1+\sqrt{13}}{2}$: the words 1001.010^ω and 210.0^ω both have value $(7+3\sqrt{13})/2$.

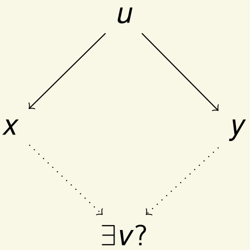
Normalisation and rewritings

Define a system of *rewriting rules* with the aim of reducing words that have the same value to a common word that would canonically represent this value.

$$210.0^\omega \leftrightarrow 202.010^\omega \leftrightarrow 1001.010^\omega$$

Confluence

Study the *confluence* of this rewriting system.



Main result

We define a way to associate with an alternate numeration system $\mathcal{N}$ a rewriting system $\mathcal{R}$, and we obtain the following result.

Proposition

The system $\mathcal{R}$ is confluent if and only if in $\mathcal{N}$, all but the last digit of every greedy representation of 1 take their maximal allowable value.