Number systems in imaginary quadratic fields

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Penney number system

Theorem (Penney, '60s)

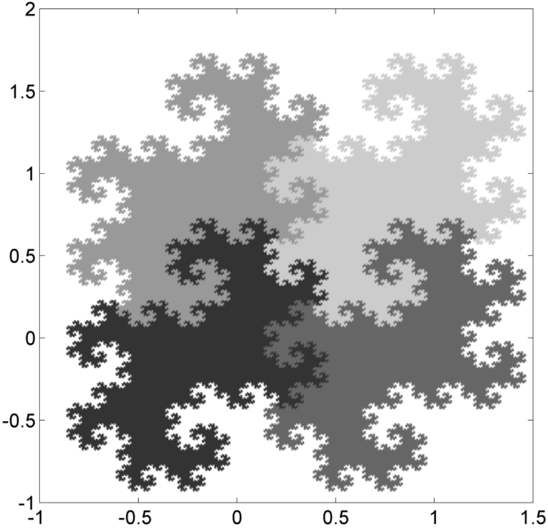
Let $\beta = -1 + \mathrm{i}$. Then every $x \in \mathbb{Z}[\mathrm{i}]$ can be uniquely written as

$$x = \sum_{j=0}^{N} \beta^j a_j, \qquad \text{where } N \in \mathbb{N}_0, a_j \in \{0, 1\}, a_N \neq 0.$$

For example, $-1 = \beta^4 + \beta^3 + \beta^2 + 1$.

On the other hand, for $\beta = 1 + \mathrm{i}$ and $\mathcal{A} = \{0, 1\}$, there is no representation of i.

This allows a representation of every complex number:



Black: Numbers with representation of the form $0 + \sum_{j=-1}^{-\infty} \beta^j a_j$ in Penney's number system.

Penney number system

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The red conditions → so-called *GNS* (with *radix* β).
E.g. $\beta = 3$, $\mathcal{A} = \{-1, 0, 1\}$ or $\beta = -2$, $\mathcal{A} = \{0, 1\}$ over $\mathbb{Z}$.

- Kátai, Szabó, '75: Characterisation of radices of *canonical* GNSs over $\mathbb{Z}[\mathrm{i}]$. ($\mathcal{A} = \{0, 1, \dots, n\}$.)
- Steidl, '89: Characterisation of radices of GNSs over $\mathbb{Z}[\mathrm{i}]$.
- Kátai, '93: The same for all $\mathcal{O}_K$ for K imaginary quadratic.

Situation in $\mathbb{Z}$:

- No GNS with radix 2.
- Radix -2: Good alphabets are $\pm\{0,1\}$.
- Good alphabets for 3: Difficult open question.
- Matula, 1978: In $\mathbb{Z}$: If $|\beta| > 2$, then β is a radix of infinitely many GNSs.

Back to imaginary quadratic fields:

Theorem (Kovács, K., '24+)

Let K be an imaginary quadratic field and $\mathcal{O} \subseteq \mathcal{O}_K$ an order. Let $\beta \in \mathcal{O}$. Then:

- *β is a radix of a GNS over $\mathcal{O}$ iff $|\beta| \neq 1$, $|1-\beta| \neq 1$.*
- *β is a radix of infinitely many GNSs over $\mathcal{O}$ iff further $|\beta| \neq 2$.*

Open question: Arbitrarily sparse alphabets?

Thank you for your attention (and for any questions)!