Number systems in imaginary quadratic fields

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## Theorem (Penney, ’60s)

Let $\beta = -1 + i$. Then every $x \in \mathbb{Z}[i]$ can be uniquely written as

$$x = \sum_{j=0}^{N} \beta^j a_j,$$

where $N \in \mathbb{N}_0$, $a_j \in \{0, 1\}$, and $a_N \neq 0$.

For example, $-1 = \beta^4 + \beta^3 + \beta^2 + 1$.

On the other hand, for $\beta = 1 + i$ and $\mathcal{A} = \{0, 1\}$, there is no representation of $i$. 

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This allows a representation of every complex number:

\[
\begin{align*}
\text{Black:} & \quad \text{Numbers with representation of the form } 0 + \sum_{j=-\infty}^{-1} \beta^j a_j \\
\text{in Penney's number system.}
\end{align*}
\]
Theorem (Penney, ’60s)

Let $\beta = -1 + i$. Then every $x \in \mathbb{Z}[i]$ can be uniquely written as

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where $N \in \mathbb{N}_0$, $a_j \in \{0, 1\}$, $a_N \neq 0$.

The red conditions $\rightarrow$ so-called GNS (with radix $\beta$).

E.g. $\beta = 3$, $\mathcal{A} = \{-1, 0, 1\}$ or $\beta = -2$, $\mathcal{A} = \{0, 1\}$ over $\mathbb{Z}$.

- Kátai, Szabó, ’75: Characterisation of radices of canonical GNSs over $\mathbb{Z}[i]$. ($\mathcal{A} = \{0, 1, \ldots, n\}$.)
- Steidl, ’89: Characterisation of radices of GNSs over $\mathbb{Z}[i]$.
- Kátai, ’93: The same for all $\mathcal{O}_K$ for $K$ imaginary quadratic.
Situation in $\mathbb{Z}$:

- No GNS with radix 2.
- Radix $-2$: Good alphabets are $\pm\{0,1\}$.
- Good alphabets for 3: Difficult open question.
- Matula, 1978: In $\mathbb{Z}$: If $|\beta| > 2$, then $\beta$ is a radix of infinitely many GNSs.

Back to imaginary quadratic fields:

**Theorem (Kovács, K., ’24+)**

Let $K$ be an imaginary quadratic field and $\mathcal{O} \subseteq \mathcal{O}_K$ an order. Let $\beta \in \mathcal{O}$. Then:

- $\beta$ is a radix of a GNS over $\mathcal{O}$ iff $|\beta| \neq 1$, $|1 - \beta| \neq 1$.
- $\beta$ is a radix of infinitely many GNSs over $\mathcal{O}$ iff further $|\beta| \neq 2$.

Open question: Arbitrarily sparse alphabets?
Thank you for your attention (and for any questions)!