Periodic unique codings of fat Sierpinski gasket

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Background

Given $\beta > 1$, let S_β be the Sierpinski gasket in $\mathbb{R}^2$ generated by the IFS:

$$f_{\alpha_0}(x) = \frac{x+\alpha_0}{\beta}, \quad f_{\alpha_1}(x) = \frac{x+\alpha_1}{\beta}, \quad f_{\alpha_2}(x) = \frac{x+\alpha_2}{\beta},$$

where $\alpha_0 = (0,0), \alpha_1 = (1,0)$ and $\alpha_2 = (0,1)$.

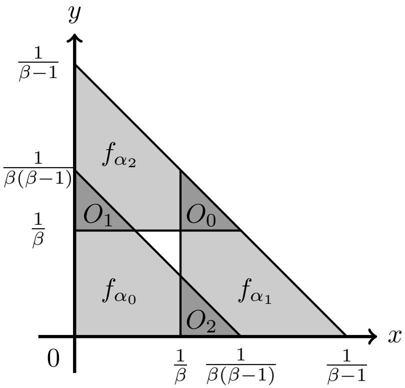

Figure: The figure of the first generation of S_β with $\beta = 18/11 \approx 1.63636$.

- If $\beta > 2$, the IFS $\{f_{\alpha_0}, f_{\alpha_1}, f_{\alpha_2}\}$ satisfies the SSC.
  If $\beta \in (1, 2)$, then the IFS $\{f_{\alpha_0}, f_{\alpha_1}, f_{\alpha_2}\}$ does not satisfy the OSC.
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- When $\beta \in (3/2, 2]$, we have

  $$\dim_H S_\beta < \frac{\log 3}{\log \beta} = \dim_S S_\beta$$

  for a dense set of $\beta \in (3/2, 2]$. (Simon and Solomyak, 2003)

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- When $\beta \in (3/2, 2]$, we also have

$$\dim_H S_\beta = \min\left\{\frac{\log 3}{\log \beta}, 2\right\}$$

  for all $\beta \in (3/2, 2]$ up to a set of zero packing dimension.
  (Hochman, 2015)

Theorem (Broomhead, Montaldi and Sidorov, 2004)

The following properties of S_β hold.

(1) *Let $\beta_* \approx 1.543686$ be the appropriate root of $\frac{1}{x^3} - \frac{1}{x^2} + \frac{1}{x} = \frac{1}{2}$. Then S_β has non-empty interior for all $\beta \in (1, \beta_*]$.*

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(3) *Let $\beta = \rho_m$ be a multinacci number, i.e., the positive root of $\frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^m} = 1$. Then*

$$\dim_H S_{\rho_m} = \dim_B S_{\rho_m} = \frac{\log \tau_m}{\log \rho_m} < \frac{\log 3}{\log \rho_m},$$

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Remark. Hasselblatt and Plante (2014) proved that S_β has non-empty interior for all

$$\beta \in [1.545, 1.5456] \cup [1.5466, 1.5485] \cup [1.5526, 1.553].$$

Open question: it is NOT known for a complete characterization of $\beta \in (1.543686, \sqrt{3})$ in which S_β has non-empty interior.

Intrinsic univoque set of S_β

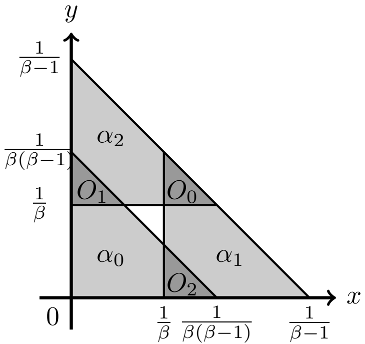
Let $\beta \in (1,2)$. For $d \in \{\alpha_0, \alpha_1, \alpha_2\}$ we define the expanding map

$$T_d(x) = \beta x - d, \quad x \in f_d(\Delta_\beta).$$

Define the intrinsic univoque set by

$$U_\beta := \left\{ x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in S_\beta : T_{d_1 \dots d_n}(x) \notin O_0 \cup O_1 \cup O_2 \ \forall n \geq 0 \right\},$$

where $T_{d_1 \dots d_n} = T_{d_n} \circ T_{d_{n-1}} \circ \cdots \circ T_{d_1}$.



Note that each $x \in U_\beta$ has a unique coding in $\{\alpha_0, \alpha_1, \alpha_2\}^{\mathbb{N}}$. Let

$$\mathbf{U}_\beta := \left\{ (d_i) \in \{\alpha_0, \alpha_1, \alpha_2\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in U_\beta \right\}.$$

Then the projection map $\pi_\beta : \mathbf{U}_\beta \to U_\beta;\ (d_i) \mapsto \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$ is bijective. Furthermore, the set-valued map $\beta \mapsto \mathbf{U}_\beta$ is non-decreasing.

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Theorem (Sidorov, 2007)

Let $\beta \in (1, 2]$. *Then*

$$\#\mathbf{U}_\beta < +\infty \quad \Longleftrightarrow \quad \beta \le \beta_G,$$

where $\beta_G \approx 1.46557$ *is a root of* $x^3 - x^2 - 1 = 0$.

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Theorem (K. and Li, 2020)

There exists a transcendental number $\beta_c \approx 1.55263$ such that

(1) *if $\beta \in (\beta_G, \beta_c)$, then $\mathbf{U}_\beta$ is countably infinite;*

(2) *if $\beta = \beta_c$, then $\mathbf{U}_\beta$ is uncountably and $\dim_H \mathbf{U}_\beta = 0$;*

(3) *if $\beta \in (\beta_c, 2)$, then $\dim_H \mathbf{U}_\beta > 0$*

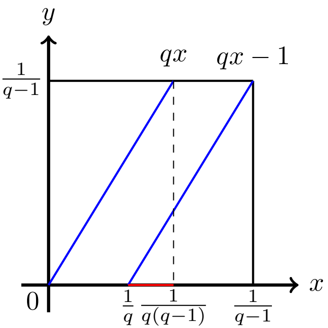
Comparison with unique q-expansion in $\mathbb{R}$

Given $q \in (1, 2]$, each $x \in [0, \frac{1}{q-1}]$ can be written as

$$x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i},$$

where the sequence $(\varepsilon_i) = \varepsilon_1 \varepsilon_2 \ldots \in \{0, 1\}^{\mathbb{N}}$ is called a q-expansion of x. Let

$$A_q := \left\{ x \in [0, \frac{1}{q-1}] : x \text{ has a unique } q - \text{expansion} \right\}.$$



For $q \in (1, 2]$ we define the symbolic univoque set

$$\mathbf{A}_q := \left\{ (\varepsilon_i) \in \{0,1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} \in A_q \right\}.$$

Then the projection map $\pi_q : \mathbf{A}_q \to A_q;\ (\varepsilon_i) \mapsto \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i}$ is bijective. Furthermore, the set-valued map $q \mapsto \mathbf{A}_q$ is non-decreasing.

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Theorem (Erdős, Joó and Komornik, 1990)

Let $q \in (1,2]$. *Then*

$$\#\mathbf{A}_q < +\infty \quad \Longleftrightarrow \quad q \le \frac{1+\sqrt{5}}{2}.$$

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Theorem (Glendinning and Sidorov, 2001)

There exists a transcendental number $q_{KL} \approx 1.78723$ (known as the Komornik-Loreti constant) such that

(1) *if $q \in (\frac{1+\sqrt{5}}{2}, q_{KL})$, then $\mathbf{A}_q$ is countably infinite;*

(2) *if $q = q_{KL}$, then A_q is uncountable and $\dim_H \mathbf{A}_q = 0$;*

(3) *if $q \in (q_{KL}, 2)$, then $\dim_H \mathbf{A}_q > 0$.*

Note that the set-valued map $q \mapsto \mathbf{A}_q$ is non-decreasing. For $k \in \mathbb{N}$ let

$$q_k := \inf \{q \in (1, 2] : \mathbf{A}_q \text{ contains a sequence of smallest period } k\}.$$

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Theorem (Allouche, Clark and Sidorov, 2009)

Each base q_k can be explicitly determined. Furthermore,

$$q_\ell > q_k \iff \ell \rhd k,$$

where $\rhd$ is the Sharkovskii order defined as

$$\begin{array}{ccccccccccc}
 & 3 & \rhd & 5 & \rhd & 7 & \rhd & \dots & \rhd & 2m+1 & \rhd & \dots \\
\rhd & 2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & \dots & \rhd & 2(2m+1) & \rhd & \dots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
\rhd & 2^n\cdot 3 & \rhd & 2^n\cdot 5 & \rhd & 2^n\cdot 7 & \rhd & \dots & \rhd & 2^n(2m+1) & \rhd & \dots \\
 & \vdots & & \vdots & & \vdots & & & & \vdots & & \\
 & & & \dots & \rhd & 8 & \rhd & 4 & \rhd & 2 & \rhd & 1.
\end{array}$$

Remark: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. If $k \rhd \ell$ in Sharkovskii ordering and if f has a point of smallest period k, then f has a point of smallest period ℓ. (Sharkovskii, 1964)

Our question

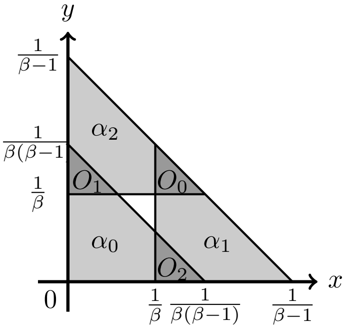
Back to the unique codings in fat Sierpinski gasket

$$\mathbf{U}_\beta = \left\{ (d_i) \in \{\alpha_0, \alpha_1, \alpha_2\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in U_\beta \right\}.$$

Note that $\beta \mapsto \mathbf{U}_\beta$ is non-decreasing. For $k \in \mathbb{N}$ we define

$$\beta_k := \inf \{\beta \in (1, 2] : \mathbf{U}_\beta \text{ contains a sequence of smallest period } k\}.$$

Question: can we determine β_k for each $k \in \mathbb{N}$?
Does the sequence (β_k) also satisfy the Sharkovskii order?



Characterization of $\mathbf{U}_\beta$

Recall that $\alpha_0 = (0,0), \alpha_1 = (1,0), \alpha_2 = (0,1)$. For $d = (d^1, d^2) \in \{\alpha_0, \alpha_1, \alpha_2\}$ we set $\overline{d^\oplus} := 1 - (d^1 + d^2)$. Then

$$d^1, d^2, \overline{d^\oplus} \in \{0,1\} \quad \text{and} \quad d^1 + d^2 + \overline{d^\oplus} = 1.$$

Proposition (K. and Li, 2020)

$(d_i) \in \mathbf{U}_\beta$ if and only if the sequences $(d_i^1), (d_i^2), (\overline{d_i^\oplus}) \in \{0,1\}^{\mathbb{N}}$ satisfy

$$c_{n+1}c_{n+2}\ldots \prec \delta(\beta) \quad \textit{if} \quad c_n = 0,$$

where $\delta(\beta) = \delta_1(\beta)d_2(\beta)\ldots$ is the quasi-greedy β-expansion of 1.

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where $\delta(\beta) = \delta_1(\beta)d_2(\beta)\ldots$ is the quasi-greedy β-expansion of 1.

Example

Let $\beta = \frac{1+\sqrt{5}}{2}$. Then $\delta(\beta) = (10)^\infty$, and

$$(\alpha_1\alpha_2\alpha_1\alpha_0\alpha_2)^\infty \sim \begin{pmatrix} d_1^1 d_2^1 \ldots d_5^1 \\ d_1^2 d_2^2 \ldots d_5^2 \\ \overline{d_1^\oplus}\,\overline{d_2^\oplus} \ldots \overline{d_5^\oplus} \end{pmatrix}^\infty = \begin{pmatrix} 10100 \\ 01001 \\ 00010 \end{pmatrix}^\infty.$$

So, $(d_i) = (\alpha_1\alpha_2\alpha_1\alpha_0\alpha_2)^\infty \in \mathbf{U}_\beta$.

Generalized Thue-Morse sequence

We define a sequence $(\mathbf{t}_n)$ of blocks in $\{0,1\}^*$. Let $\mathbf{t}_1 = 100$, and let

$$\mathbf{t}_{n+1} = \mathbf{t}_n^+ \Theta(\mathbf{t}_n^+) \quad \forall n \geq 1,$$

where the block map Θ is defined on $\Omega := \{000, 001, 100, 101\}$ by

$$\Theta : \Omega \to \Omega; \quad 000 \mapsto 101,\ 001 \mapsto 100,\ 100 \mapsto 001,\ 101 \mapsto 000.$$

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$$\begin{aligned}
\mathbf{t}_1 &= 100, \\
\mathbf{t}_2 &= 101000, \\
\mathbf{t}_3 &= 101001\,000100, \\
\mathbf{t}_4 &= 101001000101\ 000100101000.
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The sequence $(\mathbf{t}_n)$ induces a unique componentwise limit

$$(\lambda_i) = \lim_{n\to\infty} \mathbf{t}_n = 101001000101\ldots \in \{0,1\}^{\mathbb{N}}.$$

Main result

Theorem (K. and Zhang, 2024)

(1) $\beta_1 = 1$ *and* $\beta_2 = \frac{1+\sqrt{5}}{2}$.

(2) *If* $k \in 3\mathbb{N}$, *then* $k = 3(2m+1)2^n$ *for some* $m, n \in \mathbb{N}_0$, *and thus*

$$\delta(\beta_{3(2m+1)2^n}) = \begin{cases} \mathbf{t}_{n+1}^{\infty} & \text{if} \quad m = 0, \\ (\mathbf{t}_{n+2}^{+}\Theta(\mathbf{t}_{n+1}^{+})\mathbf{t}_{n+2}^{m-1})^{\infty} & \text{if} \quad m \geq 1. \end{cases}$$

Furthermore,

$$\beta_{3k} > \beta_{3\ell} \quad \Longleftrightarrow \quad k \rhd \ell \quad \text{in Sharkovskii order.}$$

(3) *If* $k = 3\ell + 1 \in 3\mathbb{N} + 1$, *then*

$$\delta(\beta_{3\ell+1}) = (101(001)^{\lfloor \frac{\ell-1}{2} \rfloor}(010)^{\lceil \frac{\ell-1}{2} \rceil}0)^{\infty}.$$

(4) *If* $k = 3\ell + 2 \in 3\mathbb{N} + 2$, *then*

$$\delta(\beta_{3\ell+2}) = (101(001)^{\ell-1}00)^{\infty}.$$

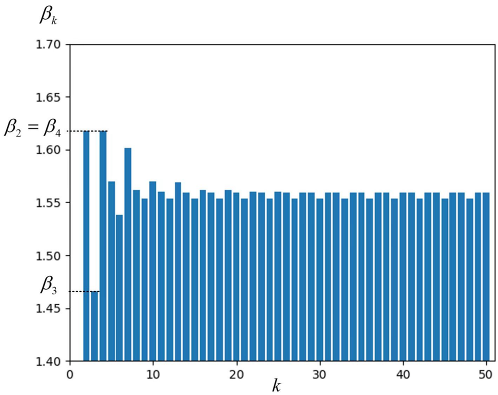
Remark

- Each β_k is a Perron number (Blanchard, 1989).
- Note that for $\ell = 2m+1$ with $m \in \mathbb{N}$ we have

$$\begin{aligned}\delta(\beta_{3(2m+1)+1}) &= (101(001)^m(010)^m0)^\infty \\ &= (101(001)^{m-1}00)^\infty = \delta(\beta_{3m+2}).\end{aligned}$$

So,

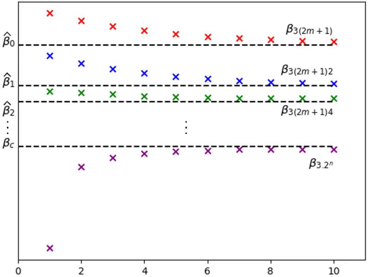
$$\beta_{6m+4} = \beta_{3m+2} \quad \text{for any } m \geq 0.$$



Asymptotic behavior of $(\beta_{3\ell})$

$$
\begin{array}{cccccccccccc}
 & \beta_{3\cdot 3} & > & \beta_{3\cdot 5} & > & \cdots & > & \beta_{3(2m+1)} & > & \cdots & > & \hat{\beta}_0 \\
> & \beta_{3\cdot 3\cdot 2} & > & \beta_{3\cdot 5\cdot 2} & > & \cdots & > & \beta_{3(2m+1)\cdot 2} & > & \cdots & > & \hat{\beta}_1 \\
> & \beta_{3\cdot 3\cdot 2^2} & > & \beta_{3\cdot 5\cdot 2^2} & > & \cdots & > & \beta_{3(2m+1)\cdot 2^2} & > & \cdots & > & \hat{\beta}_2 \\
 & \vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
> & \beta_{3\cdot 3\cdot 2^n} & > & \beta_{3\cdot 5\cdot 2^n} & > & \cdots & > & \beta_{3(2m+1)\cdot 2^n} & > & \cdots & > & \hat{\beta}_n \\
 & \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
 & \beta_c & & \beta_c & & & & \beta_c & & & & \beta_c,
\end{array}
$$

and $\beta_c > \cdots > \beta_{3\cdot 2^n} > \beta_{3\cdot 2^{n-1}} > \cdots > \beta_{3\cdot 2^2} > \beta_{3\cdot 2} > \beta_3$.



Asymptotic behavior of $(\beta_{3\ell+1})$ and $(\beta_{3\ell+2})$

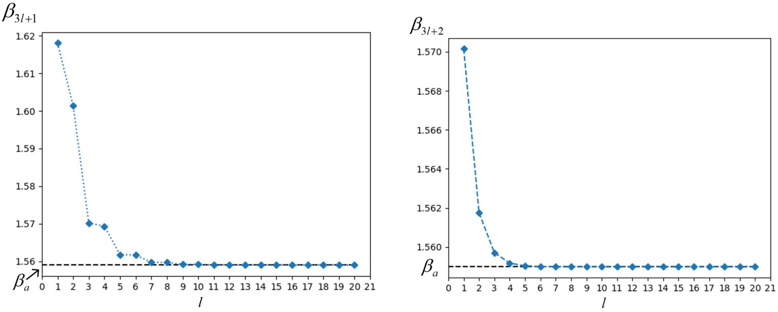


Figure: Left: the graph of $\beta_{3\ell+1}$ with $1 \leq \ell \leq 20$; right: the graph of $\beta_{3\ell+2}$ with $1 \leq \ell \leq 20$. Indeed, $\beta_{3\ell+1} \searrow \beta_a$, $\beta_{3\ell+2} \searrow \beta_a$ as $\ell \to \infty$, where $\beta_a \approx 1.55898$.

Therefore, for any $\ell \in \mathbb{N}$ we have

$$\beta_3 \leq \beta_{3\ell} \leq \beta_9 < \beta_a < \beta_{3\ell+1}, \beta_{3\ell+2} \leq \beta_2 = \frac{1+\sqrt{5}}{2},$$

where $\beta_9 \approx 1.55392$.

Lemma

For any $k \in \mathbb{N}$ *we have*

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Sketch of the proof. First we prove $\beta_2 \geq \frac{1+\sqrt{5}}{2}$. Suppose on the contrary $\beta_2 < \frac{1+\sqrt{5}}{2}$. Then $\mathbf{U}_{\frac{1+\sqrt{5}}{2}}$ contains a sequence of smallest period 2.

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$$\mathbf{U}_{\frac{1+\sqrt{5}}{2}} \ni (\alpha_0\alpha_1)^\infty \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}^\infty .$$

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For any $k \in \mathbb{N}$ we have

$$\beta_k \leq \beta_2 = \frac{1+\sqrt{5}}{2}.$$

Sketch of the proof. First we prove $\beta_2 \geq \frac{1+\sqrt{5}}{2}$. Suppose on the contrary $\beta_2 < \frac{1+\sqrt{5}}{2}$. Then $\mathbf{U}_{\frac{1+\sqrt{5}}{2}}$ contains a sequence of smallest period 2. By symmetry we assume $(d_i) = (\alpha_0\alpha_1)^\infty \in \mathbf{U}_{\frac{1+\sqrt{5}}{2}}$. Then

$$\mathbf{U}_{\frac{1+\sqrt{5}}{2}} \ni (\alpha_0\alpha_1)^\infty \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}^\infty.$$

This contradicts to the property for (d_i^1) that

$$c_{n+1}c_{n+2}\ldots \prec \delta(\frac{1+\sqrt{5}}{2}) = (10)^\infty \quad \text{if} \quad c_n = 0.$$

So, $\beta_2 \geq \frac{1+\sqrt{5}}{2}$.

Proof continu. Next we prove $\beta_k \leq \frac{1+\sqrt{5}}{2}$ for all $k \in \mathbb{N}$. Take $\beta > \frac{1+\sqrt{5}}{2}$. Then $\delta(\beta) \succ (10)^\infty$. It suffices to show that for each $k \in \mathbb{N}$ the set $\mathbf{U}_\beta$ contains a sequence of smallest period k. Easy to see that

$$\alpha_0^\infty, (\alpha_0\alpha_1)^\infty \in \mathbf{U}_\beta.$$

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- If $k = 3\ell \in 3\mathbb{N}$, then

$$\mathbf{U}_\beta \ni ((\alpha_0\alpha_1\alpha_2)^{\ell-1}\alpha_1\alpha_0\alpha_2)^\infty \sim \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{\ell-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right)^\infty .$$

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- If $k = 3\ell + 2 \in 3\mathbb{N} + 2$, then

$$\mathbf{U}_\beta \ni ((\alpha_0\alpha_1\alpha_2)^{\ell}\alpha_0\alpha_1)^\infty \sim \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{\ell} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}\right)^\infty .$$

Algorithm for a general β_k

For $k \in \mathbb{N}_{\geq 2}$ let $(d_i) = (d_1 \ldots d_k)^\infty \in \{\alpha_0, \alpha_1, \alpha_2\}^{\mathbb{N}}$ be a sequence of smallest period k. Then (d_i^1), (d_i^2) and $(\overline{d_i^{\oplus}})$ are three new periodic sequences in $\{0,1\}^{\mathbb{N}}$. Define

$$(\hat{d}_i) := \max \bigcup_{n=0}^{k-1} \left\{ \sigma^n((d_i^1)), \sigma^n((d_i^2)), \sigma^n(\overline{(d_i^{\oplus})}) \right\}.$$

Then by the characterization of $\mathbf{U}_\beta$ it follows that

$$(d_i) = (d_1 \ldots d_k)^\infty \in \mathbf{U}_\beta \quad \Longleftrightarrow \quad \delta(\beta) \succ (\hat{d}_i).$$

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Set

$$(a_i) := \min \left\{ (\hat{d}_i) : (d_i) = (d_1 \dots d_k)^\infty \text{ has smallest period } k \right\}.$$

It follows that

$$\mathbf{U}_\beta \text{ contains a sequence of smallest period } k \quad \Longleftrightarrow \quad \delta(\beta) \succ (a_i).$$

From this we can deduce that $\delta(\beta_k) = (a_i) = (a_1 \dots a_k)^\infty$.

- Upper bound: to prove $\delta(\beta_k) \preccurlyeq (\varepsilon_1 \dots \varepsilon_k)^\infty$ it suffices to show that for any $\delta(\beta) \succ (\varepsilon_1 \dots \varepsilon_k)^\infty$ the set $\mathbf{U}_\beta$ contains a sequence $(d_1 \dots d_k)^\infty$ of smallest period k.

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- Lower bound: to prove $\delta(\beta_k) \succcurlyeq (\varepsilon_1 \ldots \varepsilon_k)^\infty$, we need to show that for $\delta(\beta) = (\varepsilon_1 \ldots \varepsilon_k)^\infty$ the set $\mathbf{U}_\beta$ contains no sequence of smallest period k. This is more challenging!

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Definition

A block $a_1 \dots a_k \in \{0,1\}^*$ is called admissible if there exists an aperiodic block $d_1 \dots d_k \in \{\alpha_0, \alpha_1, \alpha_2\}^k$ such that $d_1^1 \dots d_k^1 = a_1 \dots a_k$ and

$$
\begin{aligned}
d_{j+1}^1 \dots d_k^1 d_1^1 \dots d_j^1 &\preccurlyeq a_1 \dots a_k \quad \forall 0 \le j < k, \\
d_{j+1}^2 \dots d_k^2 d_1^2 \dots d_j^2 &\preccurlyeq a_1 \dots a_k \quad \forall 0 \le j < k, \\
\overline{d_{j+1}^\oplus} \dots \overline{d_k^\oplus}\, \overline{d_1^\oplus} \dots \overline{d_j^\oplus} &\preccurlyeq a_1 \dots a_k \quad \forall 0 \le j < k.
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$$\begin{aligned} d_{j+1}^1 \dots d_k^1 d_1^1 \dots d_j^1 &\preccurlyeq a_1 \dots a_k \quad \forall 0 \le j < k, \\ d_{j+1}^2 \dots d_k^2 d_1^2 \dots d_j^2 &\preccurlyeq a_1 \dots a_k \quad \forall 0 \le j < k, \\ \overline{d_{j+1}^\oplus} \dots \overline{d_k^\oplus}\, \overline{d_1^\oplus} \dots \overline{d_j^\oplus} &\preccurlyeq a_1 \dots a_k \quad \forall 0 \le j < k. \end{aligned}$$

Difficulty: the representation block $d_1 \dots d_k$ is not necessarily unique. For example, take $a_1 \dots a_5 = 10100$. Then we have two representations

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proposition (Key proposition)

If $a_1 \dots a_{3\ell}$ is an admissible block with $\ell \geq 3$, and has a prefix

$$a_1 \dots a_9 = 101001000,$$

then $a_1 \dots a_{3\ell} \in \mathcal{B}^*(X)$, and it has a unique representation block (up to rotation) $d_1 \dots d_{3\ell} \in \{\alpha_0, \alpha_1, \alpha_2\}^{3\ell}$ satisfying

$$d_1^1 \dots d_{3\ell}^1 = a_1 \dots a_{3\ell}, \quad d_1^2 \dots d_{3\ell}^2 = (010)^\ell, \quad \overline{d_1^{\oplus}} \dots \overline{d_{3\ell}^{\oplus}} = \Theta(a_1 \dots a_{3\ell}).$$

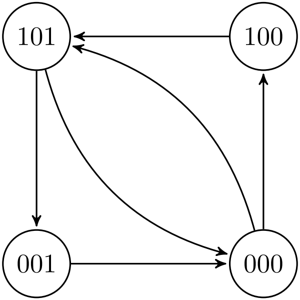

Figure: The directed graph representing the subshift of finite type X.

Proof of the key proposition

Since $a_1 \ldots a_{3\ell}$ is admissible, it has a representation $d_1 \ldots d_{3\ell} \in \{\alpha_0, \alpha_1, \alpha_2\}^{3\ell}$. Here we only prove

$$a_1 \ldots a_9 \in \mathcal{B}^*(X) \quad \text{and} \quad d_1^2 \ldots d_9^2 = (010)^3.$$

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Note that

$$d_1^1 \ldots d_9^1 = a_1 \ldots a_9 = 101001000 \in \mathcal{B}^*(X),$$

$$d_i^1 + d_i^2 + \overline{d_i^{\oplus}} = 1 \quad \forall 1 \le i \le 9.$$

Suppose $d_1^2 \ldots d_9^2 \succcurlyeq \overline{d_1^{\oplus}} \ldots \overline{d_9^{\oplus}}$. By the definition of admissibility and using $a_1 \ldots a_9 = 101001000$ it follows that

$$11, \quad 10101, \quad 10100101 \quad \text{and} \quad 101001001$$

are all forbidden in $d_1^1 \ldots d_{3\ell}^1$, $d_1^2 \ldots d_{3\ell}^2$ and $\overline{d_1^{\oplus} \ldots d_{3\ell}^{\oplus}}$. From this we can deduce that (needs explanation)

$$\begin{pmatrix} d_1^1 \ldots d_9^1 \\ d_1^2 \ldots d_9^2 \\ \overline{d_1^{\oplus}} \ldots \overline{d_9^{\oplus}} \end{pmatrix} = \begin{pmatrix} 101 & 001 & 000 \\ 010 & 010 & 010 \\ 000 & 100 & 101 \end{pmatrix}.$$

The proof can be proceeded by induction. For this we also need the following inequalities of generalized Thue-Morse sequence. Recall that

$$\delta(\beta_c) = \lambda_1\lambda_2\ldots = 101001000101\ldots.$$

Let $(\gamma_i) = \Theta(\lambda_1\lambda_2\ldots) = 000100101000\ldots$. Then for any $n \geq 0$ we have

$$\gamma_1\ldots\gamma_{3\cdot 2^n - i} \prec \lambda_{i+1}\ldots\lambda_{3\cdot 2^n} \preccurlyeq \lambda_1\ldots\lambda_{3\cdot 2^n - i},$$
$$\gamma_1\ldots\gamma_{3\cdot 2^n - i} \preccurlyeq \gamma_{i+1}\ldots\gamma_{3\cdot 2^n} \prec \lambda_1\ldots\lambda_{3\cdot 2^n - i}$$

for all $0 \leq i < 3\cdot 2^n$.

Final remarks and future work

The main result can be adapted to a general class of fat Sierpinski gasket in $\mathbb{R}^2$ generated by

$$\{f_1(x) = \lambda x + \mathbf{p}_1, \quad f_2(x) = \lambda x + \mathbf{p}_2, \quad f_3(x) = \lambda x + \mathbf{p}_3\},$$

where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are non-colinear vectors in $\mathbb{R}^2$.

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where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are non-colinear vectors in $\mathbb{R}^2$.

- Extend to more general planar self-similar sets with overlaps, and consider the associated periodic points.
- Extend to higher dimensional fat Sierpinski gasket or self-similar sets with overlaps.
- Study one parameter family of open dynamical systems, and determine the critical parameter in which the open dynamical system contains a point of smallest period k.

Thank you!