Geometry of restricted continued fraction digit sets and Lüroth digit sets

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– joint work with M. Keßeböhmer –

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Restricted continued fraction digits

For $\Lambda \subset \mathbb{N} := \{1, 2, \ldots\}$ let

$$F_\Lambda := \{[a_1, a_2, \ldots] \mid \forall n \in \mathbb{N}; a_n \in \Lambda\}$$

with the continued fraction expansion

$$[a_1, a_2, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Kesseböhmer & Zhou (2006) showed the Texan Conjecture:

$$\{\dim_H(F_\Lambda) \mid \Lambda \subset \mathbb{N}\} = [0, 1],$$

i. e.

$$\forall x \in [0, 1] \quad \exists \Lambda \subset \mathbb{N} \quad \text{s. t. } \dim_H(F_\Lambda) = x.$$

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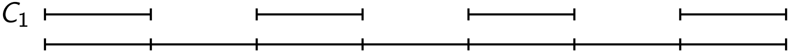

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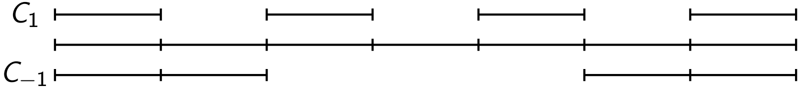

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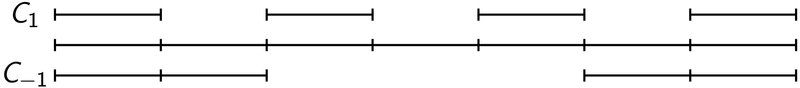

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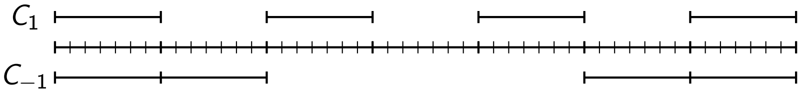

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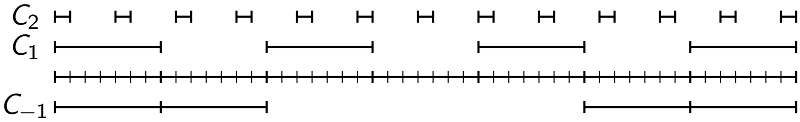

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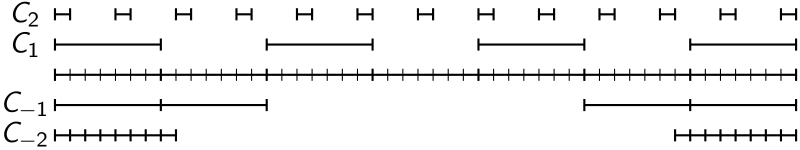

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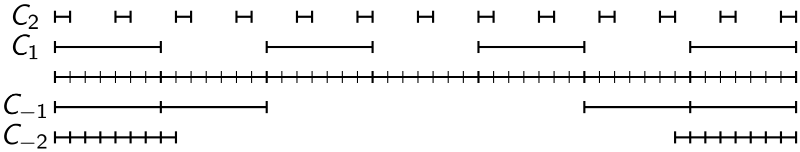

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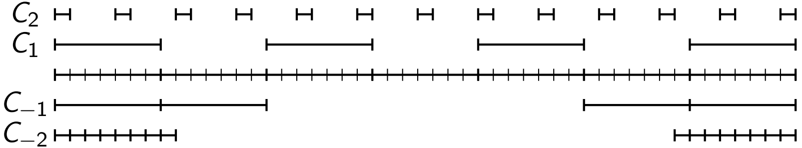
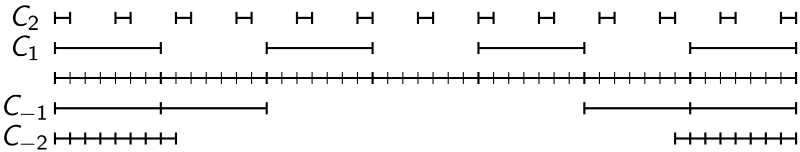


$$\dim_{H,M}(C_{\pm 1}) = \frac{\log 4}{\log 7} = \frac{\log 16}{\log 49} = \dim_{H,M}(C_{\pm 2})$$

- Cantor set C_k will look more and more like $[0,1]$ for $k \to \infty$
- Cantor set C_{-k} will look more and more like $\{0,1\}$ for $k \to \infty$
- C_k and C_{-k} differ significantly in their gap structure

Minkowski content

- proposed as measure of lacunarity for fractals, Mandelbrot '82: *"a fractal is to be called lacunar if its gaps tend to be large, in the sense that they include large intervals (discs, or balls)."*



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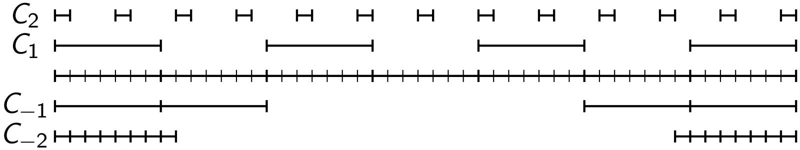

Definition (Minkowski content of $F \subset \mathbb{R}$ for which $\dim_M(F)$ exists)

$$\mathcal{M}(F) := \lim_{\varepsilon \to 0} \varepsilon^{\dim_M(F)-1} |F_\varepsilon| \quad \text{if } \lim \text{ exists}$$

$$\overline{\mathcal{M}}(F) := \limsup_{\varepsilon \to 0} \varepsilon^{\dim_M(F)-1} |F_\varepsilon| \quad \textit{upper Minkowski content}$$

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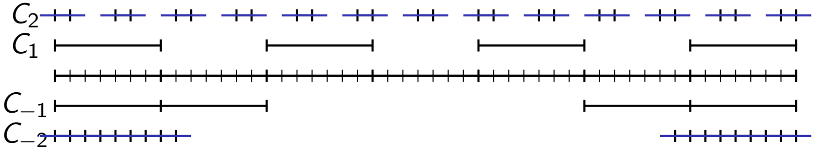

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Minkowski content – restricted continued fraction digit sets

For $\Lambda \subset \mathbb{N}$ with $\#\Lambda \geq 2$

$$F_\Lambda := \{[a_1, a_2, \ldots] \mid \forall n \in \mathbb{N};\ a_n \in \Lambda\}$$

- $D := \dim_H(F_\Lambda) < 1 \quad \Rightarrow \Lambda \neq \mathbb{N}$
- μ equilibrium measure
- h_μ its measure theoretical entropy.
- For simplicity assume that for $k \in \mathbb{N} \setminus \Lambda$ we have $k \pm 1 \in \Lambda \cup \{0\}$ (guarantees: cIFS is strongly regular):

Theorem

$\mathcal{M}(F_\Lambda)$ exists, is positive and finite and

$$\mathcal{M}(F_\Lambda) = \frac{2^{1-D}}{(1-D)h_\mu} \lim_{m\to\infty} \sum_{a\in\mathbb{N}\setminus\Lambda} \sum_{|\omega|=m} \left|\Phi_\omega([a])\right|^D .$$

Restricted Lüroth digits

For $\Lambda \subset \mathbb{N}$ with $\#\Lambda \geq 2$ consider the IFS on the unit interval:

$$\Psi := \{\psi_n : x \mapsto -a_n x + t_n \mid n \in \Lambda\} \quad \text{with}$$

$$a_n := \zeta(s)^{-1} \frac{1}{n^s}, \quad t_n := \zeta(s)^{-1} \sum_{k=n}^{\infty} \frac{1}{k^s}, \; n \in \mathbb{N}$$

for fixed $s > 1$ with the Riemann zeta-function ζ.

Ψ is an (infinitely generated) IFS of linear maps

L_Λ: Invariant set of Ψ

= set of all Lüroth expansions omitting the digits from $\mathbb{N} \setminus \Lambda$

$\dim_{H,M}(L_\Lambda)$ is the unique real $\delta > 0$ for which

$$\sum_{k \in \Lambda} \frac{1}{k^{\delta s}} = \zeta(s)^{\delta}.$$

Minkowski content – restricted Lüroth digits

Theorem (non-lattice)

If $\{\log a_n \mid n \in \Lambda\}$ *does* **not** *generate a discrete subgroup of* $\mathbb{R}$ *(system is non-lattice) then* $\mathcal{M}(L_\Lambda)$ *exists, is positive and finite and*

$$\mathcal{M}(L_\Lambda) = \frac{2^{1-\delta}(\zeta(s\delta)/\zeta(s)^\delta - 1)}{(1-\delta)h_\mu}$$

Example (lattice)

Fix $\ell \in \mathbb{N}_{\geq 2}$, $s > 1$ with

$$\frac{s \log \ell}{\log \zeta(s)} \in \mathbb{Q}.$$

If $\Lambda \subset \{\ell^k \mid k \in \mathbb{N}\}$, then for some $q \in \mathbb{N}$

$$\{\log a_n \mid n \in \Lambda\} \subset \{ks \log \ell - \log(\zeta(s)) \mid k \in \mathbb{N}\} \subset \log(\zeta(s))/q\mathbb{Z}$$

The Minkowski content of L_Λ does not exist.

Proof idea – Renewal theory

- F_Λ invariant under the (infinitely generated) conformal IFS on the unit interval:
$$\Phi := \left\{ \phi_k : x \mapsto \frac{1}{x+k} \mid k \in \Lambda \right\},$$
i. e. $F_\Lambda = \bigcup_{k \in \Lambda} \phi_k(F_\Lambda)$.

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