Another proof of finiteness of monochromatic arithmetic progressions in the Fibonacci word

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Van der Waerden’s theorem

A *monochromatic arithmetic progression* (MAP) is a set of positions of the same symbol with a constant distance $d$. 

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>8</th>
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<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
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</tr>
</tbody>
</table>

$\text{length}(\text{MAP}) = 3$. 

Theorem (Baudet’s conjecture, 1927)

For any $A$, and some $k$, $N \in \mathbb{N}$, a MAP of length $k$ must exist in a string of length $N$. 

MAPs in $F$ are finite.
A monochromatic arithmetic progression (MAP) is a set of positions of the same symbol with a constant distance $d$. Take $d = 2$ and start at position 3 $\Rightarrow$ length(MAP) = 3.

Theorem (Baudet’s conjecture, 1927)

For any $A$, and some $k, N \in \mathbb{N}$, a MAP of length $k$ must exist in a string of length $N$. 
Finiteness of MAPs

Question

Does the Fibonacci word $f$ have an infinite MAP for some $d'$?

Answer: No.
Finiteness of MAPs

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Proofs so far:

- **Durand-Goyheneche, 2018**: $\exists$ irrational dynamical eigenvalue of the subshift generated by $x$, iff $x$ admits an infinite MAP.

- **Aedo-Grimm, 2021 (unpublished)**: Based on the theory of model sets.
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We prove it using the definition of $f$ as a rotation sequence.
Plot for \( n \geq 1, \langle n\tau \rangle := n\tau \mod 1 \).

\[
f : 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ \cdots
\]

**Figure:** Generating the Fibonacci word.
Rotation sequence

Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

$$f : 0 1 0 0 1 0 1 0 0 1 0 0 1 0 1 0 1 0 \cdots$$

Figure: Generating the Fibonacci word.
Rotation sequence

Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

$$f : \begin{array}{cccccccccccccc}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots
\end{array}$$

Figure: Generating the Fibonacci word.
Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

$f : 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ \ldots$

**Figure:** Generating the Fibonacci word.
Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

$$f : 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad \cdots$$

\[0 \quad \langle 2\tau \rangle \quad \tau^{-2} \quad \langle 4\tau \rangle \quad \langle \tau \rangle \quad \langle 3\tau \rangle \quad 1\]

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Rotation sequence

Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

\[
f : 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ \ldots
\]

**Figure:** Generating the Fibonacci word.
Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

$f : 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ \cdots$

**Figure:** Generating the Fibonacci word.
Rotation sequence

Plot for \( n \geq 1, \langle n \tau \rangle := n \tau \mod 1. \)

\[ f : 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ \cdots \]

**Figure:** Generating the Fibonacci word.

**Proposition**

*The Fibonacci word contains no infinite MAPs.*
Rotation sequence

Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

$$f : 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ \cdots$$

**Figure**: Generating the Fibonacci word.

**Proposition**

The Fibonacci word contains no infinite MAPs.

**Proof.**

the orbit of $\langle n\tau \rangle$ for all $n$ on $[0, 1]$ interval is dense. This extends to any one-sided Sturmian sequence for the same reason.