Another proof of finiteness of monochromatic arithmetic progressions in the Fibonacci word

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Van der Waerden's theorem

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	⋯
f :	0	1	0	0	1	0	1	0	0	1	0	0	1	0	1	0	⋯

A monochromatic arithmetic progression (MAP) is a set of positions of the same symbol with a constant distance d.

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A monochromatic arithmetic progression (MAP) is a set of positions of the same symbol with a constant distance d.

Take $d = 2$ and start at position 3 $\implies$ length(MAP) $= 3$.

Theorem (Baudet's conjecture, 1927)

For any $\mathcal{A}$, and some $k, N \in \mathbb{N}$, a MAP of length k must exist in a string of length N.

Finiteness of MAPs

Question

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Answer: No.

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- Durand-Goyheneche, 2018: $\exists$ irrational dynamical eigenvalue of the subshift generated by $\mathbf{x}$, iff $\mathbf{x}$ admits an infinite MAP.
- Aedo-Grimm, 2021 (unpublished): Based on the theory of model sets.

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We prove it using the definition of **f** as a rotation sequence.

Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

$\mathbf{f}$: 0 1 0 0 1 0 1 0 0 1 0 0 1 0 1 0 $\cdots$

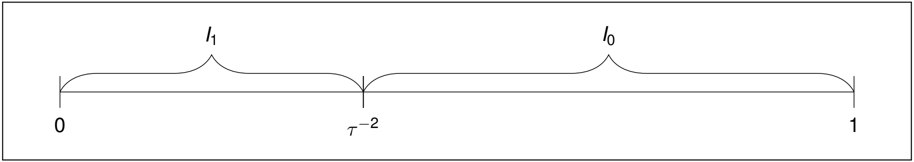
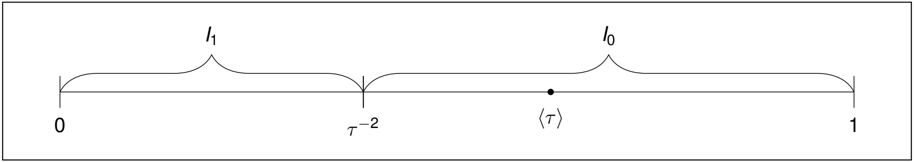


Figure: Generating the Fibonacci word.

Rotation sequence

Plot for $n \geq 1$, $\langle n\tau \rangle := n\tau \mod 1$.

$\mathbf{f}: \mathbf{0}$ 1 0 0 1 0 1 0 0 1 0 0 1 0 1 0 $\cdots$



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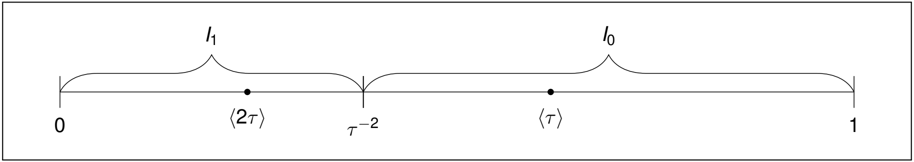

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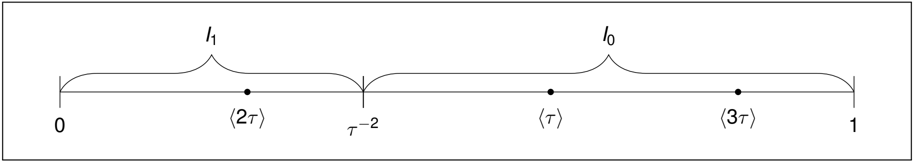

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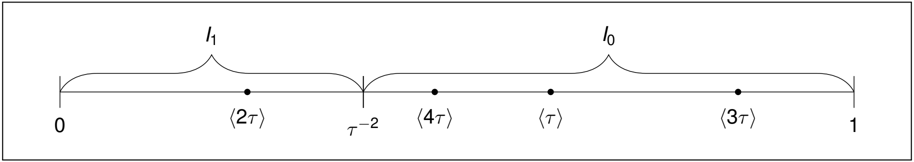

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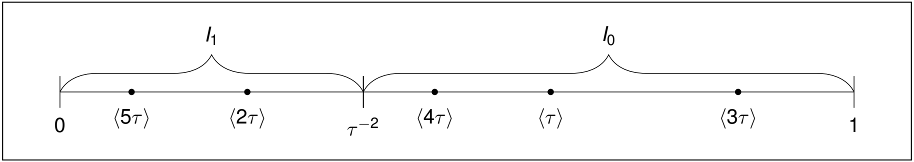

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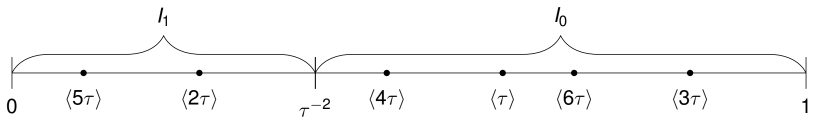

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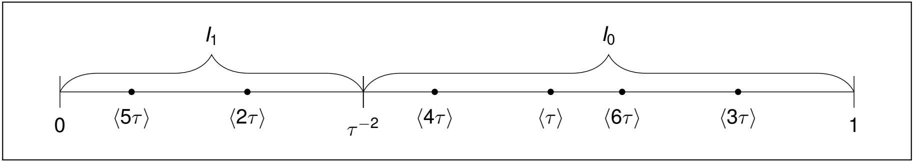

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Proposition

The Fibonacci word contains no infinite MAPs.

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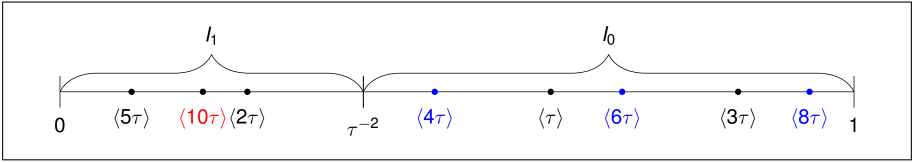

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Proof.

the orbit of $\langle n\tau \rangle$ for all n on $[0, 1]$ interval is dense. This extends to any one-sided Sturmian sequence for the same reason. ∎