Collisions of digit sums in two bases

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Introduction

We are interested in the joint behaviour of the sum-of-digits function in two coprime bases p and q. In particular, how many solutions does the equation

$$s_p(n) = s_q(n)$$

have? We call a number n that satisfies the equation above a *collision* (with respect to the bases p and q).

Main results

Theorem 1 (J, 2024)**.** *Let p, q be two coprime integers, let $r > 0$ be a rational number. Then there exist infinitely many integers n such that*

$$s_p(n) = rs_q(n).$$

In particular there exists a constant $c = c(p, q, r) > 0$, such that

$$\#\{0 \leq n \leq N : s_p(n) = rs_q(n)\} \gg N^c.$$

For the case $r = 1$ we get the following corollary for collisions:

Corollary 2 (J, 2024)**.** *Let $1 < p < q$ be two coprime integers. Then*

$$\{0 \leq n \leq N : s_p(n) = s_q(n)\} \gg N^{c-\varepsilon},$$

where c is determined by

$$\lim_{N\to\infty} \frac{\log\left(\#\left\{0 \leq n \leq N : s_q(n)) = \left\lfloor \frac{p-1}{2} \log_p(N) \right\rfloor \right\}\right)}{\log(N)} = c.$$

Previous results

Theorem 3 (de la Bretèche, Stoll, Tennenbaum, 2019)**.** *Let $p, q > 1$ be two multiplicatively independent integers. Then*

$$\{s_p(n)/s_q(n) : n \geq 1\}$$

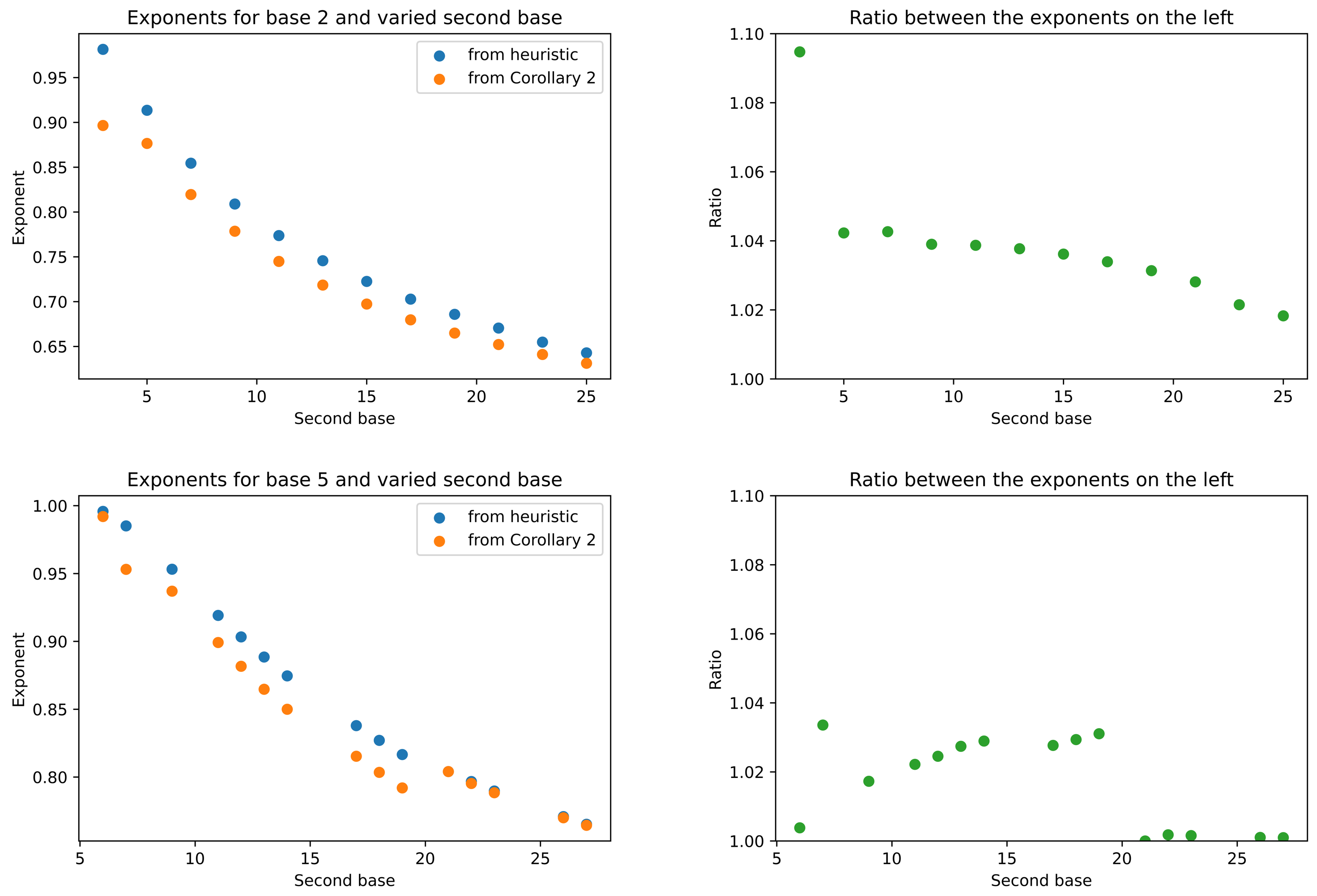
is dense in $\mathbb{R}^+$.

Theorem 4 (Spiegelhofer, 2021)**.** *For all $\varepsilon > 0$, we have*

$$\#\{0 \leq n \leq N : s_2(n) = s_3(n)\} \gg N^{\log 3/\log 4 - \varepsilon}.$$

Comparison to heuristic results

We compare the exponent given by Corollary 2 to numerical calculations (implementing a heuristic). One base is fixed at 2 resp. 5 and the second base is varied.



The difference in the exponents tends to 0 (resp. their ratio tends to 1) as the difference between the two bases tends to infinity.

Methology

The proofs of Theorem 1 and Theorem 4 rely on the following three steps:

1. Find a d such that for almost all $nd < N$, we have that
$$|s_p(dn) - rs_q(dn)| < \log(N)^{1/2} \log\log(N)^{1/2+\varepsilon}.$$
2. Show that the pair $(s_p(dn), s_q(dn))$ is equidistributed modulo $m = \log(N)^{1/2}/\log\log(N)^5$ in the above interval.
3. We use a trick, such that for each n that satisfies Step 1 and $s_p(dn) \equiv s_q(dn) \equiv 0 \mod m$, we can find an a such that $s_p(dn + a) = rs_q(dn + a)$.

The proof of Theorem 1 relies on the improvement of Steps 1 and 2, to allow d to be of size up to $N^{1-\varepsilon}$.

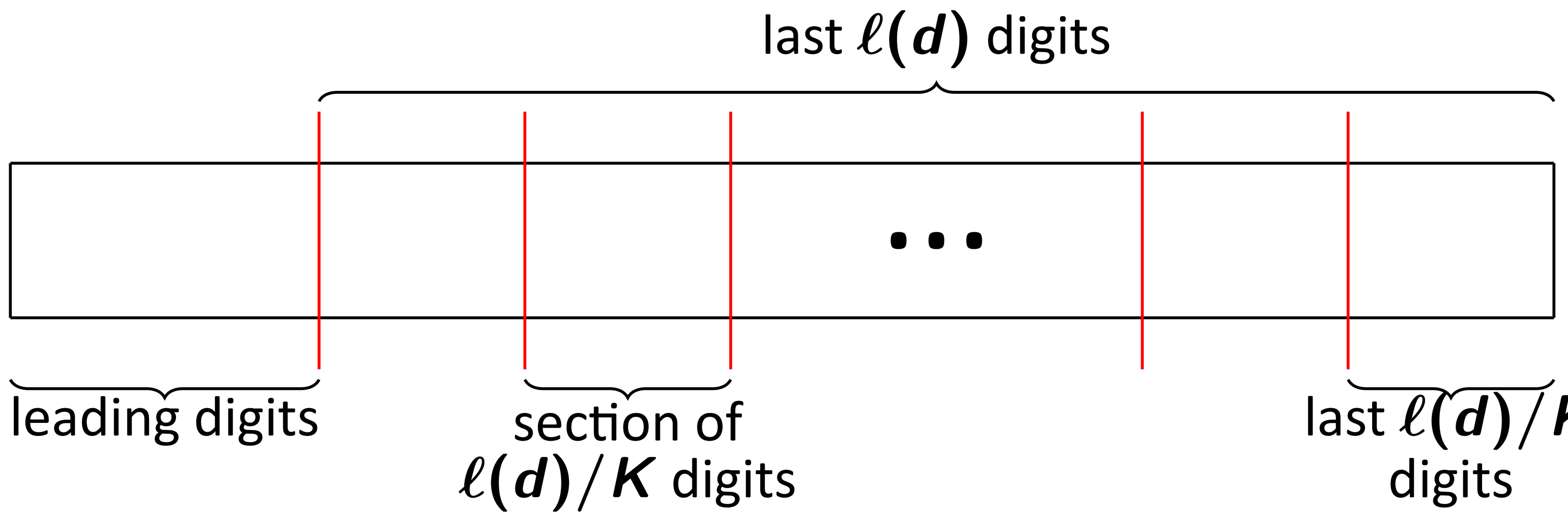
Arithmetic progressions with large step size

Proposition 5. *Let $q > 1$ be an integer. Let $d > 0$ be an integer that has no long strings of 0 or $p-1$. Then almost all $n \leq N/d$ satisfy*

$$\left|s_q(dn) - \frac{q-1}{2}\log_q(N)\right| < \log(N)^{1/2} \log\log(N)^{1/2+\varepsilon}$$

and furthermore the set $\{s_q(dn) : 1 \leq dn \leq N\}$ is equidistributed modulo every $m < \log(N)^{1/2}/\log\log(N)^5$.

The key idea in the proofs of both statements above is to divide the digits of each number into $K + 1$ sections: K equally big sections up to the length $\ell(d)$ of d and one section for the remaining digits. The digits of nd are split in following way:



The first and the last section have the expected value of the sum-of-digits function by classical results. Each middle section is a Beatty sequence modq^a, which we treat with new methods.
For the second statement, we follow recent proofs regarding the level of distribution of the sum-of-digits function, but with the added difficulty that m is dependent on the size N.

Open problems

1. Is it possible to prove that the exponent from the heuristic is correct?
2. Does Theorem 1 generalise to multiplicatively independent bases?
3. Can we find a tuple $(p_0, p_1, \ldots, p_k)$, such that $s_{p_0}(n) = \cdots = s_{p_k}(n)$ has only finitely many solutions?
4. Can Theorem 1 and Corollary 2 be generalised to more than 2 bases?

References

[1] Régis de la Bretèche, Thomas Stoll, and Gérald Tenenbaum. Somme des chiffres et changement de base. *Annales de l'Institut Fourier*, 69:2507–2518, 01 2019.

[2] Lukas Spiegelhofer. Collisions of digit sums in bases 2 and 3. *Israel Journal of Mathematics*, pages 1–28, 2021.