Collisions of digit sums in two bases

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Motivation

- If $p < q$ are positive integers, we expect $s_p(n) < s_q(n)$.
- How many n satisfy $s_p(n) = s_q(n)$?
- Which ratios $s_p(n)/s_q(n)$ can be attained?
- How often are they attained?

Previous Results

Theorem (de la Bretèche, Stoll, Tennenbaum, 2019)

Let $p, q > 1$ be two multiplicatively independent integers. Then

$$\{s_p(n)/s_q(n) : n \geq 1\}$$

is dense in $\mathbb{R}^+$.

Theorem (Spiegelhofer, 2021)

There exist infinitely many n such that

$$s_2(n) = s_3(n).$$

Methodology

1. Find a d such that for almost all $nd < N$, we have that

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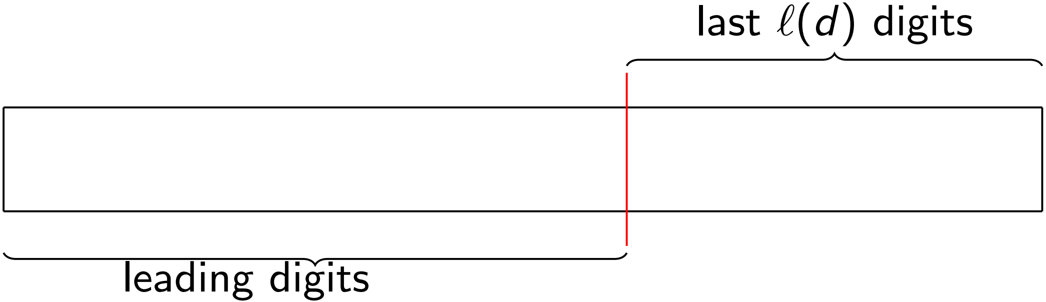
3. We use a trick, such that for each n that satisfies Step 1 and $s_p(nd) \equiv s_q(nd) \equiv 0 \mod m$, we can find an a such that $s_p(nd + a) = s_q(nd + a)$.

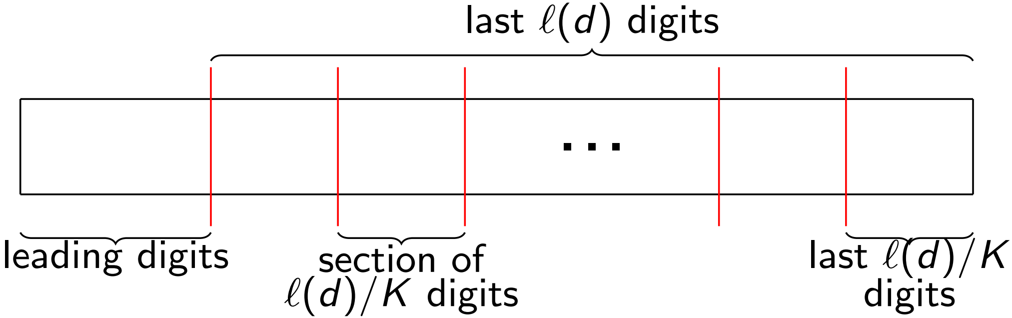
Methodology

- Assume $d \leq \sqrt{N}$
- Divide digits of each nd into 2 sections:



Improvements

- Assume $d \leq N^{1-\varepsilon}$ is generic
- Divide digits of each nd into $K+1$ sections:



New Results

Theorem (J., 2024)

Let p, q be two coprime integers, let $r > 0$ be a rational number. Then there exist infinitely many integers n such that

$$s_p(n) = rs_q(n).$$

In particular there exists a constant $c = c(p, q, r) > 0$, such that

$$\#\{0 \leq n \leq N : s_p(n) = rs_q(n)\} \gg N^c.$$

New Results

Corollary (J., 2024)

Let $1 < p < q$ *be two coprime integers. Then*

$$\{0 \leq n \leq N : s_p(n) = s_q(n)\} \gg N^{c-\varepsilon},$$

where c *is determined by*

$$\lim_{N\to\infty} \frac{\log\left(\#\left\{0 \leq n \leq N : s_q(n)) = \left\lfloor \frac{p-1}{2} \log_p(N)\right\rfloor\right\}\right)}{\log(N)} = c.$$

Open Questions

1. Does Theorem 3 generalise to multiplicatively independent bases?
2. Can Theorem 3 and Corollary 4 be generalised to more than 2 bases?
3. Can we find a tuple $(p_0, p_1, \ldots, p_k)$, such that $s_{p_0}(n) = \cdots = s_{p_k}(n)$ has only finitely many solutions?