Collisions of digit sums in two bases

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Motivation

▶ If $p < q$ are positive integers, we expect $s_p(n) < s_q(n)$.

▶ How many $n$ satisfy $s_p(n) = s_q(n)$?

▶ Which ratios $s_p(n)/s_q(n)$ can be attained?

▶ How often are they attained?
Previous Results

Theorem (de la Bretèche, Stoll, Tennenbaum, 2019)

Let $p, q > 1$ be two multiplicatively independent integers. Then

$$\left\{ \frac{s_p(n)}{s_q(n)} : n \geq 1 \right\}$$

is dense in $\mathbb{R}^+$.

Theorem (Spiegelhofer, 2021)

There exist infinitely many $n$ such that

$$s_2(n) = s_3(n).$$
Methodology

1. Find a $d$ such that for almost all $nd < N$, we have that

$$|s_p(nd) - s_q(nd)| < \log(N)^{1/2} \log \log(N)^{1/2+\varepsilon}.$$
Methodology

1. Find a $d$ such that for almost all $nd < N$, we have that

$$|s_p(nd) - s_q(nd)| < \log(N)^{1/2} \log \log(N)^{1/2 + \varepsilon}.$$ 

2. Show that the pair $(s_p(nd), s_q(nd))$ is equidistributed modulo

$m = \log(N)^{1/2} / \log \log(N)^5$ in the above interval.
Methodology

1. Find a $d$ such that for almost all $nd < N$, we have that

$$|s_p(nd) - s_q(nd)| < \log(N)^{1/2} \log \log(N)^{1/2+\varepsilon}.$$ 

2. Show that the pair $(s_p(nd), s_q(nd))$ is equidistributed modulo $m = \log(N)^{1/2} / \log \log(N)^5$ in the above interval.

3. We use a trick, such that for each $n$ that satisfies Step 1 and $s_p(nd) \equiv s_q(nd) \equiv 0 \mod m$, we can find an $a$ such that $s_p(nd + a) = s_q(nd + a)$. 
Methodology

- Assume $d \leq \sqrt{N}$
- Divide digits of each $nd$ into 2 sections:
  
  - last $\ell(d)$ digits
  - leading digits
Improvements

- Assume $d \leq N^{1-\varepsilon}$ is generic
- Divide digits of each $nd$ into $K + 1$ sections:

  - Last $\ell(d)$ digits
  - Leading digits
  - Section of $\ell(d)/K$ digits
  - Last $\ell(d)/K$ digits
New Results

Theorem (J., 2024)

Let \( p, q \) be two coprime integers, let \( r > 0 \) be a rational number. Then there exist infinitely many integers \( n \) such that

\[
s_p(n) = rs_q(n).
\]

In particular there exists a constant \( c = c(p, q, r) > 0 \), such that

\[
\#\{0 \leq n \leq N : s_p(n) = rs_q(n)\} \gg N^c.
\]
Corollary (J., 2024)

Let $1 < p < q$ be two coprime integers. Then

$$\{0 \leq n \leq N : s_p(n) = s_q(n)\} \gg N^{c-\varepsilon},$$

where $c$ is determined by

$$\lim_{N \to \infty} \frac{\log \left( \# \left\{ 0 \leq n \leq N : s_q(n) = \left\lfloor \frac{p-1}{2} \log_p(N) \right\rfloor \right\} \right)}{\log(N)} = c.$$
Open Questions

1. Does Theorem 3 generalise to multiplicatively independent bases?
2. Can Theorem 3 and Corollary 4 be generalised to more than 2 bases?
3. Can we find a tuple \((p_0, p_1, \ldots, p_k)\), such that \(s_{p_0}(n) = \cdots = s_{p_k}(n)\) has only finitely many solutions?