Thermodynamic Formalism of Self Similar Overlapping Measures

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Introduction

In studying fractal geometry there have been many questions that one seeks to answer, primarily amongst these questions are those that relate to Hausdorff dimension. Many of the most known results of this type require some assumption on the separation of the fractal in question. We begin to consider fractal measures supported on subsets of $\mathbb{R}$. Amongst the folklore of these measures it is understood that the structure of these measures can be understood in terms of the types of overlap that occur, in particular so called exact overlaps. Therefore we seek a way to count the growth rate of exact overlaps and use this to calculate a bound for the dimension of the measures.

Overview

We will show how classical results from thermodynamic formalism can be applied to an individual self similar overlapping measure to understand the dimension of the measure. Beyond this we shall see how the thermodynamic formalism of certain lines of rational slope correspond to the dynamics of exact overlaps in self similar overlapping measures. We restrict our attention to rational slope due to an identification of self similar overlapping measures with projections of well studied fractals and results from the study of these projections.

The collection of measures that we consider could be considered to be the simplest case of measures where these questions are in some sense unanswered. The case of measures with two intervals with interval has been solved by [1] and our measures are the simplest overlap systems with three intervals.

IFS, Exact Overlaps and Projections

We define an Iterated Function System (IFS) as a collection of contracting maps, $\{\phi_1, \cdots, \phi_n\}$.

For a finite words $a, b \in \{1, \cdots, j\}^n$. We can define $\phi_a = \phi_{a(1)}\phi_{a(2)} \dots \phi_{al(n)}$. a and b are said to exactly overlap if $\phi_a = \phi_b$.

This can be seen as finding two compositions of contractions which result in the same action on a space. We particularly care about this for given an exact overlap identical extensions of these exactly overlapping words will result in the same map, as such will affect growth rate.

We are looking at systems of self similar overlapping measures. The family of measures that we are concerned with are measures that can be arrived at through orthogonal projections of the Sierpinski's Gasket along lines of rational slope. Formally, let $C = \{(\frac{x}{2}, \frac{y}{2}), (\frac{x+1}{2}, \frac{y}{2}), (\frac{x}{2}, \frac{y+1}{2})\}$ be an IFS, and S be the unique closed compact invariant set of C acting on $\mathbb{R}^2$. This set comes with a natural measure on it, the $(1/3, 1/3, 1/3)$ Bernoulli measure, denoted ν, which corresponds to choosing the maps in C with equal probability at every stage of construction.

Define $P_m(x, y) = x + my$, $m \in \mathbb{Q}$. Rather than look at $P_m(S)$, as this is simply the line $[0, 1]$, we define our measure as the measure μ such that $P_{m*}\nu = \mu$. Equivalently, for $A \subset [0, 1]$, $\mu(A) = \nu(\{(x, y) : P_m(x, y) \in A\})$. This measure can be visualised as the measure made by successively concentrating mass across the intervals $[0, 1/2), [1/2, 1), [m/2, m/2 + 1/2)$.

Restriction to rational parameters

By applying Marstrand's projection theorem [2] to our projections P_m, we know that there is a full measure set where no dimension drop occurs. From this one can show that the full measure set is irrational values of m and as such we restrict to rational m.

Recoverability Automata

Let $\underline{x}, \underline{y} \in \{0, 1, m\}^n$, $n \in \mathbb{N}, m \in \mathbb{Q}, 0 < m < 1$. Then we define the recoverability function $R_n(\underline{x}, \underline{y}) = \sum_{i=0}^{n} 2^{n-i}(\underline{x}(i) - \underline{y}(i))$.

We call a pair of words recoverable if $|R_n(\underline{x}, \underline{y})| < 1$. A pair of words is called irrecoverable if it is not recoverable. We notice due to the doubling in R_n and maximal difference in digit sums, when a pair of words has become irrecoverable it will remain irrecoverable for all possible extensions.

We note the importance of this definition for a pair of words being recoverable implies the existence of pair of extensions which lead to exact overlaps.

From this recoverability function we define a Recoverability Automata (S_R, T_R). This automata has states of all possible values of R_n for a fixed $m \in \mathbb{Q}$ and labeled directed transition edges given by all possible differences $R_{n+1}(\underline{x}, \underline{y}) - R_n(\underline{x}, \underline{y})$, labeled by the pair $(\underline{x}(n+1), \underline{y}(n+1))$ to get the difference. By construction the language accepted by this automata is that of all recoverable pairs of words.

Pressure of Given Overlaps

The largest eigenvalue, λ, of the transition matrix for the recoverability automata for the rational value m can provide a lower bound for the dimension drop in the self similar fractal measure that occurs as $P_{m*}\nu$.

We note that this is an application of the Bowen-Perron-Frobenius theorem as the recoverability automata is 'strongly connected' and as such yields an irreducible non-negative matrix. Therefore we may conclude the topological pressure of the overlaps is λ.

Invariant Lattices

While we have shown that thermodynamic formalism can deal with a single rational parameter, we seek to further this by being able to consider multiple parameters at once. To do this we create a series of lattices which are in some sense invariant under the choice of m. We do this by considering lattices which expresses in two dimensional binary expansions the possible values of α, β where we express the possible values of R_n as $\alpha p - \beta q$ for $m = p/q$. Let $\Sigma_B = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}^{\mathbb{N}}$, $a \in \Sigma_B$. The potential function $\phi : \Sigma_B^n \to \mathbb{R}$, for specific non-negative integer matrices A_i counts the growth rate of paths through this lattice. We define,

$$\phi(a) = \limsup_{n\to\infty} \log \left(\frac{(1, 1, 1, 1)A_{a_1 \dots a_n}(1, 0, 0, 0)'}{(1, 1, 1, 1)A_{a_2 \dots a_n}(1, 0, 0, 0)'} \right)$$

For a fixed $m = p/q$ we can define a map $\tau_m : S_R \to \Sigma_B^n$. For words of a fixed length, i.e the language of $\{0, 1, m\}^n$, the map τ is surjective.

Toral Equivalence

Let $\Sigma_B^n = \{0, 1, m\}^n$ be the approximation of Σ_B by factors of length n. We define a line of rational slope, $l_{p/q}$, $\gcd(p, q) = 1, p, q \in \mathbb{N}$ as $l_{p/q} = \{(x', y') : x \equiv x' \mod (1) , y \equiv y' \mod (1) , mx = y\}$. Finally we define the map $\rho : \Sigma_B^n \to \mathbb{R}^2$ by $\rho(\underline{x}) = \sum_{i=1}^{n} 2^{-i}\underline{x}(i)$.

This equivalence $\mod (1)$ can be seen as viewing $\rho(\Sigma_B)$ as a torus rather than plane, justifying this sections title.

One can then show that $l_{p/q}$ and all the points in $(x, y) \in \Sigma_B^n$ such that $\min d_2(\rho(x, y), l_{p/q}) \leq 2^n$ are precisely the points which correspond to words that are recoverable.

This is to say that as the length of the words in Σ_B^n increases, the line $l_{p/q}$ counts all the words that cause exact overlaps and none more.

Main Result - Correspondence of Pressures

The topological pressure $P(\phi)$, of a function ϕ, with i_k in an alphabet A, $\omega \in \Sigma_A$ and σ the standard left shift is defined as,

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1 \dots i_n} \exp \left(\sup_{\omega \in [i_1 \dots i_n]} \sum_{j=0}^{n-1} \phi(\sigma^j \omega) \right) \right)$$

For the potential function as above, the topological pressure function $P(\phi)$ (restricted to lines of rational slope $l_{p/q}$) can provide a lower bound for the dimension drop caused by exact overlaps.

Further Work

There are two clear extensions for this line of work. The first of these is to refine the main result of this work - this is to say find some manner to gain a deeper understanding of the pressure function and its behaviour over ranges of rational projections. A hope would be to be able to show that this pressure tends to 0 for most rational projections.

Other than this I have begun to generalise this approach to systems with a greater number of overlaps and to systems with a greater number of base intervals with the aim of extending this to a system of an algebraic base, most likely a Pisot number.

References

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[2] J. M. Marstrand.
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