Analysis of Regular Sequences: Summatory Functions and Divide-and-Conquer Recurrences

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Prelude 1: Find Min & Max via Divide & Conquer

- Partition into two sets of (almost) equal size;
- Find min and max in both parts individually & recursively;
- Compare minima, compare maxima.

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In other words:

$$\begin{aligned} M(2n) &= 2M(n) + 2, \\ M(2n+1) &= M(n) + M(n+1) + 2. \end{aligned}$$

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Matrix–Vector Form: Sum of Digits

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q-Regular Sequences

Vector-valued sequence $v\colon \mathbb{N}_0 \to \mathbb{C}^D$ with

$$v(qn + r) = A_r v(n)$$

for all $0 \le r < q$ and $n \ge 0$.

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Constants:

- $q \ge 2$, $D \ge 1$: integers;
- $A_0, \ldots, A_{q-1}$: $D \times D$-matrices.

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First component of v: *q-regular sequence* (Allouche–Shallit 1992).

Analysis of Regular Sequences

Theorem (Dumas 2013; H–Krenn 2020)

- *$x(n)$: q-regular sequence, first component of $v(n)$*

$$\sum_{0\le n<N} x(n) = \sum_{\substack{\lambda\in\sigma(C)\\ |\lambda|>R}} N^{\log_q \lambda} \sum_{0\le k<m_C(\lambda)} \frac{(\log N)^k}{k!}\,\Phi_{\lambda k}(\{\log_q N\}) + O\Big(N^{\log_q R}(\log N)^{\max\{m_C(\lambda)\colon |\lambda|=R\}}\Big)$$

as $N\to\infty$, where $\Phi_{\lambda k}$ are suitable 1-periodic functions.

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- *$\sigma(C)$: spectrum of C*

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- *$m_C(\lambda)$: size of the largest Jordan block of C associated with λ*

$$\sum_{0\le n<N} x(n) = \sum_{\substack{\lambda\in\sigma(C)\\ |\lambda|>R}} N^{\log_q \lambda} \sum_{0\le k<m_C(\lambda)} \frac{(\log N)^k}{k!} \Phi_{\lambda k}(\{\log_q N\})$$

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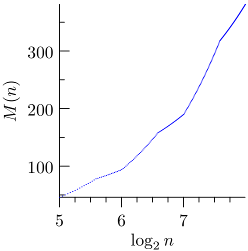
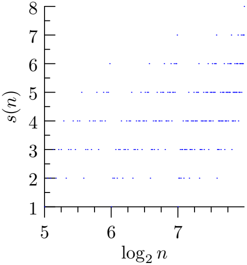
$$\Phi_{\lambda k}(u) = \sum_{\mu \in \mathbb{Z}} \varphi_{\lambda k \mu} \exp(2\mu\pi i u)$$

- Fourier coefficients can be computed numerically (using a functional equation for the corresponding Dirichlet series)

Original Sequence, Summatory Function, Renormalisation





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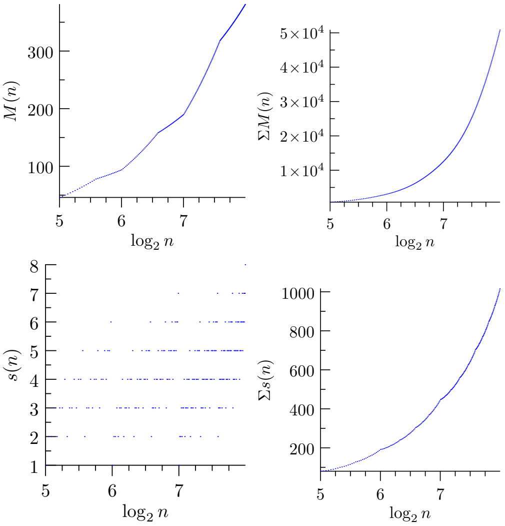

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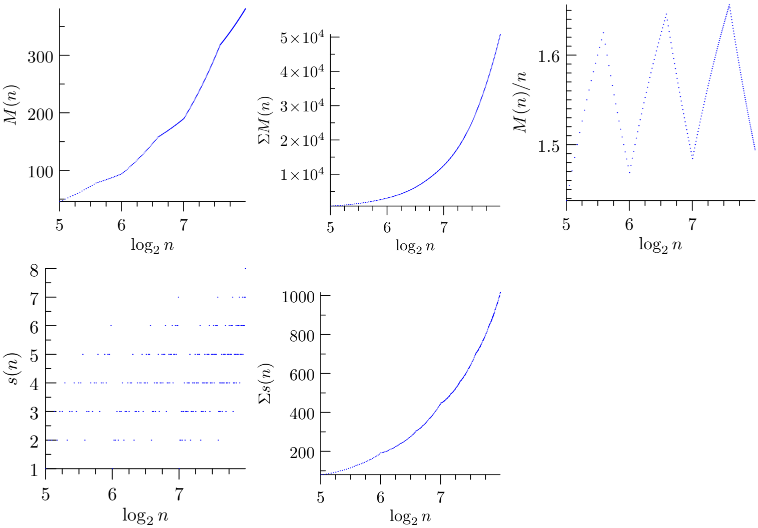

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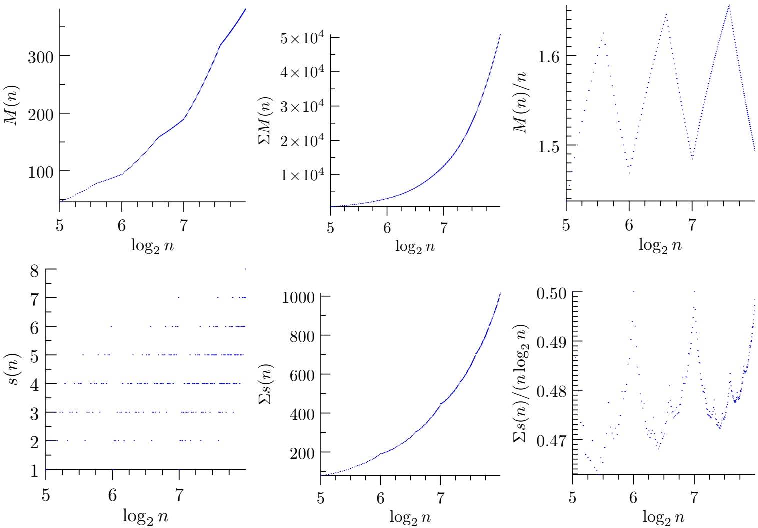

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Spoilers

1. Yes (almost always)
2. For divide & conquer sequences

Iterated Summatory Function

$$\Sigma v(N) := \sum_{0 \le n < N} v(n)$$

Theorem (H.–Krenn–Lechner 2024)

Let x be a q-regular sequence with matrices $(A_r)_{0 \le r < q}$. Set $C := \sum_{0 \le r < q} A_r$. Assume that C has an eigenvalue $\neq 0$. Then there is a non-negative integer k such that $\Sigma^k x$ admits a "good asymptotic expansion".

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$$\Sigma v(qN + r) = \sum_{0 \le qn + r' < qN + r} v(qn + r')$$

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$$\begin{aligned} \Sigma v(qN + r) &= \sum_{0 \le qn + r' < qN + r} v(qn + r') \\ &= \sum_{0 \le n < N} \sum_{0 \le r' < q} v(qn + r') + \sum_{0 \le r' < r} v(qN + r') \end{aligned}$$

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$$\Sigma v(qN + r) = \sum_{0 \le n < N} \sum_{0 \le r' < q} v(qn + r') + \sum_{0 \le r' < r} v(qN + r')$$

$$= \sum_{0 \le n < N} \sum_{0 \le r' < q} A_{r'} v(n) + \sum_{0 \le r' < r} A_{r'} v(N)$$

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Repeating: asymptotic domination by $D^k C$ (if C has an eigenvalue $\neq 0$).

Divide-and-Conquer Sequences

Divide-and-conquer sequence:

$$x(n) = \alpha x\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \beta x\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n)$$

for $n \geq 2$ (α, β given positive constants, g given function ("toll function"), $x(1)$ given).

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Theorem (Hwang–Janson–Tsai 2023)

Assume that there is an $\varepsilon > 0$ such that $g(n) = O(n^{\log_2(\alpha+\beta)-\varepsilon})$. Then

$$x(n) = n^{\log_2(\alpha+\beta)}\Phi(\{\log_2 n\}) + O(n^{\log_2(\alpha+\beta)-\varepsilon})$$

for $n \to \infty$ where Φ is a continuous, 1-periodic function.

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Question: Relation to "our" result?

Divide-and-Conquer and Regular Sequences

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Divide-and-Conquer and Regular Sequences

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$$\begin{aligned} x(2n) &= (\alpha + \beta)x(n) + g(2n) \\ x(2n+1) &= \alpha x(n) + \beta x(n+1) + g(2n+1) \end{aligned}$$

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If g is regular, then x is regular.

Summatory Function of the Forward Difference

$$x(N) = x(0) + \sum_{0 \le n < N} (x(n+1) - x(n))$$

Summatory Function of the Forward Difference

$$x(N) = x(0) + \sum_{0 \le n < N} (x(n+1) - x(n))$$

- Forward difference of regular sequence is regular
- Summatory function of regular sequence is regular

... but can we say something in general?

Forward Difference of Divide-and-Conquer Sequence

Set $\Delta x(n) := x(n+1) - x(n)$.

Forward Difference of Divide-and-Conquer Sequence

Set $\Delta x(n) := x(n+1) - x(n)$.
Consider divide-and-conquer sequence

$$\begin{aligned} x(2n) &= (\alpha+\beta)x(n) + g(2n) \\ x(2n+1) &= \alpha x(n) + \beta x(n+1) + g(2n+1) \end{aligned}$$

Forward Difference of Divide-and-Conquer Sequence

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$\Rightarrow$

$$\begin{aligned} \Delta x(2n) &= \beta \Delta x(n) + g(2n+1) - g(2n) \\ \Delta x(2n+1) &= \alpha \Delta x(n) + g(2n+2) - g(2n+1). \end{aligned}$$

Forward Difference of Divide-and-Conquer Sequence

Set $\Delta x(n) \coloneqq x(n+1) - x(n)$.
Consider divide-and-conquer sequence

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Dimension 1 (plus dimension of linear representation of g): particulary simple.

Divide-and-Conquer: Result

Theorem (H.–Krenn–Lechner 2024)

Let x be a divide-and-conquer sequence with polynomial toll function of degree $k \geq 1$. Then (for 1-periodic continuous functions Φ and Ψ and $n \to \infty$):

Divide-and-Conquer: Result

Theorem (H.–Krenn–Lechner 2024)

Let x be a divide-and-conquer sequence with polynomial toll function of degree $k \geq 1$. Then (for 1-periodic continuous functions Φ and Ψ and $n \to \infty$):

- **Case 1a.** If $\alpha + \beta > 2^k$ and $2^k > \max\{\alpha, \beta\}$, then

$$x(n) = n^{\log_2(\alpha+\beta)}\Phi(\{\log_2 n\}) + n^k \Psi(\{\log_2 n\}) + O(n^{\log_2 \max\{\alpha,\beta\}}).$$

Divide-and-Conquer: Result

Theorem (H.–Krenn–Lechner 2024)

Let x be a divide-and-conquer sequence with polynomial toll function of degree $k \geq 1$. Then (for 1-periodic continuous functions Φ and Ψ and $n \to \infty$):

- **Case 1b.** If $\alpha + \beta > 2^k$ and $\max\{\alpha, \beta\} \geq 2^k$, then

$$x(n) = n^{\log_2(\alpha+\beta)}\Phi(\{\log_2 n\}) + O(n^{\log_2 \max\{\alpha,\beta\}}(\log n)^{[\max\{\alpha,\beta\}=2^k]}).$$

Divide-and-Conquer: Result

Theorem (H.–Krenn–Lechner 2024)

Let x be a divide-and-conquer sequence with polynomial toll function of degree $k \geq 1$. Then (for 1-periodic continuous functions Φ and Ψ and $n \to \infty$):

- **Case 2.** If $\alpha + \beta = 2^k$, then

$$x(n) = n^k (\log n)\Phi(\{\log_2 n\}) + n^k \Psi(\{\log_2 n\}) + O\big(n^{\log_2 \max\{\alpha,\beta\} + [\alpha=\beta]\varepsilon}\big)$$

for any $\varepsilon > 0$.

Divide-and-Conquer: Result

Theorem (H.–Krenn–Lechner 2024)

Let x be a divide-and-conquer sequence with polynomial toll function of degree $k \geq 1$. Then (for 1-periodic continuous functions Φ and Ψ and $n \to \infty$):

- **Case 3.** If $2^k > \alpha + \beta > 2^{k-1}$, then

$$x(n) = n^k \Phi(\{\log_2 n\}) + n^{\log_2(\alpha+\beta)} \Psi(\{\log_2 n\}) + O\Big(n^{\log_2 \max\{\alpha,\beta,2^{k-1}\} + [\max\{\alpha,\beta\}=2^{k-1}]\varepsilon} \times (\log n)^{[\max\{\alpha,\beta\}<2^{k-1}]}\Big)$$

for any $\varepsilon > 0$.

Divide-and-Conquer: Result

Theorem (H.–Krenn–Lechner 2024)

Let x be a divide-and-conquer sequence with polynomial toll function of degree $k \geq 1$. Then (for 1-periodic continuous functions Φ and Ψ and $n \to \infty$):

- **Case 4.** If $2^{k-1} \geq \alpha + \beta$, then

$$x(n) = n^k \Phi(\{\log_2 n\}) + O(n^{k-1}(\log n)^E),$$

where

$$E := 1 + [\alpha + \beta = 2^{k-1}]([k \geq 2 \text{ and } c_{k-1} \neq 0] + [k = 1 \text{ and } d_0 + d_1 \neq 0])$$

with

$$d_0 := (1-\beta)x(1) - g(1) + g(0), \quad d_1 := g(1) - (1-\beta)x(1).$$

Min-Max: Fourier Coefficients

Computing Fourier coefficients using the general result ...