Analysis of Regular Sequences: Summatory Functions and Divide-and-Conquer Recurrences

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Prelude 1: Find Min & Max via Divide & Conquer

- Partition into two sets of (almost) equal size;
- Find min and max in both parts individually & recursively;
- Compare minima, compare maxima.

Number $M(n)$ of comparisons when finding min and max of $n$ elements:

$$M(n) = M(\lceil n/2 \rceil) + M(\lfloor n/2 \rfloor) + 2.$$  
In other words:

$$M(2n) = 2M(n) + 2,$$
$$M(2n+1) = M(n) + M(n+1) + 2.$$
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Prelude 2: Binary Sum of Digits

- Binary expansion $n = \sum_{j \geq 0} \varepsilon_j 2^j$
- Sum of digits $s(n) = \sum_{j \geq 0} \varepsilon_j$

In other words:

$$s(2n) = s(n)$$
$$s(2n + 1) = s(n) + 1$$
Prelude 2: Binary Sum of Digits

- Binary expansion \( n = \sum_{j \geq 0} \varepsilon_j 2^j \)
- Sum of digits \( s(n) = \sum_{j \geq 0} \varepsilon_j \)

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s(n) = s(\lfloor n/2 \rfloor) + [n \text{ is odd}].
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$$s(n) = s(\lfloor n/2 \rfloor) + [n \text{ is odd}].$$

In other words:

$$s(2n) = s(n),$$
$$s(2n + 1) = s(n) + 1.$$
Matrix–Vector Form: Sum of Digits

Recall:

\[ s(2n) = s(n), \]
\[ s(2n + 1) = s(n) + 1. \]
Matrix–Vector Form: Sum of Digits

Recall:

\[ s(2n) = s(n), \]
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Consider

\[ v(n) := \begin{pmatrix} s(n) \\ 1 \end{pmatrix}. \]
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Recall:

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Consider

\[ \mathbf{v}(n) := \begin{pmatrix} s(n) \\ 1 \end{pmatrix}. \]

Then

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Consider

\[ \nu(n) := \begin{pmatrix} s(n) \\ 1 \end{pmatrix}. \]

Then

\[ \nu(2n) = \begin{pmatrix} s(2n) \\ 1 \end{pmatrix} = \begin{pmatrix} s(n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nu(n), \]

.
Matrix–Vector Form: Sum of Digits

Recall:

\[ s(2n) = s(n), \]
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\[ v(n) := \begin{pmatrix} s(n) \\ 1 \end{pmatrix}. \]

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\[ v(2n) = \begin{pmatrix} s(2n) \\ 1 \end{pmatrix} = \begin{pmatrix} s(n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v(n), \]
\[ v(2n + 1) = \begin{pmatrix} s(2n + 1) \\ 1 \end{pmatrix}. \]
Matrix–Vector Form: Sum of Digits

Recall:

\[
\begin{align*}
    s(2n) &= s(n), \\
    s(2n + 1) &= s(n) + 1.
\end{align*}
\]

Consider

\[
v(n) := \begin{pmatrix} s(n) \\ 1 \end{pmatrix}.
\]

Then

\[
\begin{align*}
    v(2n) &= \begin{pmatrix} s(2n) \\ 1 \end{pmatrix} = \begin{pmatrix} s(n) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v(n), \\
    v(2n + 1) &= \begin{pmatrix} s(2n + 1) \\ 1 \end{pmatrix} = \begin{pmatrix} s(n) + 1 \\ 1 \end{pmatrix}.
\end{align*}
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Matrix–Vector Form: Find Min and Max

Recall:

\[ M(2n) = 2M(n) + 2, \]
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Matrix–Vector Form: Find Min and Max

Recall:

\[ M(2n) = 2M(n) + 2, \]
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Consider

\[ v(n) := (M(n) \quad M(n + 1) \quad 1)^{\top}. \]
Matrix–Vector Form: Find Min and Max

Recall:

\[ M(2n) = 2M(n) + 2, \]
\[ M(2n + 1) = M(n) + M(n + 1) + 2. \]

Consider

\[ v(n) := (M(n) \quad M(n + 1) \quad 1)^T. \]

Then

\[ v(2n) = \begin{pmatrix} M(2n) \\ M(2n + 1) \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} v(n), \]
Matrix–Vector Form: Find Min and Max

Recall:

\[ M(2n) = 2M(n) + 2, \]
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Consider

\[ \nu(n) := (M(n) \quad M(n + 1) \quad 1)^T. \]

Then

\[ \nu(2n) = \begin{pmatrix} M(2n) \\ M(2n + 1) \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \nu(n), \]
\[ \nu(2n + 1) = \begin{pmatrix} M(2n + 1) \\ M(2n + 2) \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \nu(n). \]
Vector-valued sequence \( v : \mathbb{N}_0 \to \mathbb{C}^D \) with

\[
v(qn + r) = A_r v(n)
\]

for all \( 0 \leq r < q \) and \( n \geq 0 \).
Vector-valued sequence $\mathbf{v} : \mathbb{N}_0 \rightarrow \mathbb{C}^D$ with

$$\mathbf{v}(qn + r) = A_r \mathbf{v}(n)$$

for all $0 \leq r < q$ and $n \geq 0$.

Constants:

- $q \geq 2, D \geq 1$: integers;
- $A_0, \ldots, A_{q-1}: D \times D$-matrices.
Vector-valued sequence $v: \mathbb{N}_0 \rightarrow \mathbb{C}^D$ with

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Constants:

- $q \geq 2$, $D \geq 1$: integers;
- $A_0, \ldots, A_{q-1}$: $D \times D$-matrices.

Theorem (Dumas 2013; H–Krenn 2020)

\( x(n) \): q-regular sequence, first component of \( v(n) \)

\[
\sum_{0 \leq n < N} x(n) = \sum_{\lambda \in \sigma(C)} N^{\log_q \lambda} \sum_{0 \leq k < m_C(\lambda)} \frac{(\log N)^k}{k!} \Phi_{\lambda k}(\{\log_q N\})
\]

\[ + O(\sum_{|\lambda| = R} N^{\log_q R} (\log N)^{\max\{m_C(\lambda): |\lambda|=R\}}) \]

as \( N \to \infty \), where \( \Phi_{\lambda k} \) are suitable 1-periodic functions.
Theorem (Dumas 2013; H–Krenn 2020)

- \( x(n) \): \( q \)-regular sequence, first component of \( v(n) \)
- \( C := A_0 + \cdots + A_{q-1} \)
- \( \sigma(C) \): spectrum of \( C \)

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\sum_{0 \leq n < N} x(n) = \sum_{\lambda \in \sigma(C)} N^{\log_q \lambda} \sum_{0 \leq k < m_C(\lambda)} \frac{(\log N)^k}{k!} \Phi_{\lambda k}(\{\log_q N\})
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\[+ O(N^{\log_q R} (\log N)^{\max\{m_C(\lambda) : |\lambda| = R\}})\]

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Theorem (Dumas 2013; H–Krenn 2020)

- \( x(n) \): \( q \)-regular sequence, first component of \( \nu(n) \)
- \( C := A_0 + \cdots + A_{q-1} \)
- \( \sigma(C) \): spectrum of \( C \)
- \( R := \lim_{k \to \infty} \sup \{ \| A_{r_1} \cdots A_{r_k} \|^{1/k} \mid 0 \leq r_1, \ldots, r_k < q \} \): Joint spectral radius of \( A_0, \ldots, A_{q-1} \)

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\sum_{0 \leq n < N} x(n) = \sum_{\lambda \in \sigma(C)} N^{\log_q \lambda} \sum_{0 \leq k < m_C(\lambda)} \frac{(\log N)^k}{k!} \Phi_{\lambda k}(\{\log_q N\}) \\
\quad + O \left( N^{\log_q R} (\log N)^{\max\{m_C(\lambda) : |\lambda|=R\}} \right)
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as \( N \to \infty \), where \( \Phi_{\lambda k} \) are suitable \( 1 \)-periodic functions.
Analysis of Regular Sequences

Theorem (Dumas 2013; H–Krenn 2020)

- $x(n)$: $q$-regular sequence, first component of $v(n)$
- $C := A_0 + \cdots + A_{q-1}$
- $\sigma(C)$: spectrum of $C$
- $R := \lim_{k \to \infty} \sup \{ \| A_{r_1} \cdots A_{r_k} \|^{1/k} : 0 \leq r_1, \ldots, r_k < q \}$: Joint spectral radius of $A_0, \ldots, A_{q-1}$
- $m_C(\lambda)$: size of the largest Jordan block of $C$ associated with $\lambda$

$$
\sum_{0 \leq n < N} x(n) = \sum_{\lambda \in \sigma(C)} N^{\log_q \lambda} \sum_{0 \leq k < m_C(\lambda)} \frac{(\log N)^k}{k!} \Phi_{\lambda k}(\{\log_q N\}) + O(N^{\log_q R} (\log N)^{\max\{m_C(\lambda) : |\lambda| = R\}})
$$

as $N \to \infty$, where $\Phi_{\lambda k}$ are suitable 1-periodic functions.
For $|\lambda| > R$ and $0 \leq k < m_C(\lambda)$:
Analysis of Regular Sequences: Fluctuations

For $|\lambda| > R$ and $0 \leq k < m_C(\lambda)$:

- $\Phi_{\lambda k}$ is Hölder continuous
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- Pointwise convergence of the Fourier series

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\Phi_{\lambda k}(u) = \sum_{\mu \in \mathbb{Z}} \varphi_{\lambda k \mu} \exp(2\mu \pi i u)
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For $|\lambda| > R$ and $0 \leq k < m_C(\lambda)$:

- $\Phi_{\lambda k}$ is Hölder continuous
- Pointwise convergence of the Fourier series

$$\Phi_{\lambda k}(u) = \sum_{\mu \in \mathbb{Z}} \varphi_{\lambda k \mu} \exp(2\mu \pi i u)$$

- Fourier coefficients can be computed numerically (using a functional equation for the corresponding Dirichlet series)
Original Sequence, Summatory Function, Renormalisation
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\[ M(n) \]

\[ \Sigma M(n) \]

\[ s(n) \]

\[ \Sigma s(n) \]
Original Sequence, Summatory Function, Renormalisation

\[ M(n) = \sum_{\log_2 n}^{100} \frac{1}{n} \times 10^4 \]

\[ \sum M(n) = \sum_{\log_2 n}^{200} \frac{1}{n} \times 10^4 \]

\[ M(n)/n = \sum_{\log_2 n}^{300} \frac{1}{n} \times 10^4 \]

\[ s(n) = \sum_{\log_2 n}^{400} \frac{1}{n} \times 10^4 \]

\[ \sum s(n) = \sum_{\log_2 n}^{600} \frac{1}{n} \times 10^4 \]

\[ \sum \left( \frac{1}{n} \times 10^4 \right) = 1000 \]
Original Sequence, Summatory Function, Renormalisation

\[
\begin{align*}
M(n) &\quad \text{log}_2 n \\
\Sigma M(n) &\quad \text{log}_2 n \\
M(n)/n &\quad \text{log}_2 n \\
s(n) &\quad \text{log}_2 n \\
\Sigma s(n) &\quad \text{log}_2 n \\
\Sigma s(n)/(n \log_2 n) &\quad \text{log}_2 n
\end{align*}
\]
Observations and Questions

Observations

- Some regular sequences are “smooth enough” such that an asymptotic formula makes sense.
- Other regular sequences are not “smooth enough”; taking the summatory function might help.

Questions

1. Does taking the summatory function a finite number of times always lead to a smooth asymptotic behaviour?
2. Can we say something about classes where the original sequence is smooth enough?

Spoilers

1. Yes (almost always)
2. For divide & conquer sequences
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1. Does taking the summatory function a finite number of times always lead to a smooth asymptotic behaviour?
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Spoilers

1. Yes (almost always)
2. For divide & conquer sequences
Iterated Summatory Function

\[ \sum v(N) := \sum_{0 \leq n < N} v(n) \]

**Theorem (H.–Krenn–Lechner 2024)**

Let \( x \) be a \( q \)-regular sequence with matrices \((A_r)_{0 \leq r < q}\). Set 
\( C := \sum_{0 \leq r < q} A_r \). Assume that \( C \) has an eigenvalue \( \neq 0 \).

Then there is a non-negative integer \( k \) such that \( \Sigma^k x \) admits a “good asymptotic expansion”.

Iterated Summatory Function

\[ \Sigma v(N) := \sum_{0 \leq n < N} v(n) \]

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\[ \Sigma v(qN + r) \]
Iterated Summatory Function

$$\Sigma v(N) := \sum_{0 \leq n < N} v(n)$$

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Let $x$ be a $q$-regular sequence with matrices $(A_r)_{0 \leq r < q}$. Set $C := \sum_{0 \leq r < q} A_r$. Assume that $C$ has an eigenvalue $\neq 0$.

Then there is a non-negative integer $k$ such that $\Sigma^k x$ admits a “good asymptotic expansion”.

$$\Sigma v(qN + r) = \sum_{0 \leq qn + r' < qN + r} v(qn + r')$$
Iterated Summatory Function

\[ \Sigma v(N) := \sum_{0 \leq n < N} v(n) \]

**Theorem (H.–Krenn–Lechner 2024)**

Let \( x \) be a \( q \)-regular sequence with matrices \( (A_r)_{0 \leq r < q} \). Set \( C := \sum_{0 \leq r < q} A_r \). Assume that \( C \) has an eigenvalue \( \neq 0 \).

Then there is a non-negative integer \( k \) such that \( \Sigma^k x \) admits a “good asymptotic expansion”.

\[ \Sigma v(qN + r) = \sum_{0 \leq qn + r' < qN + r} v(qn + r') \]

\[ = \sum_{0 \leq n < N} \sum_{0 \leq r' < q} v(qn + r') + \sum_{0 \leq r' < r} v(qN + r') \]
Iterated Summatory Function

\[ \Sigma v(N) := \sum_{0 \leq n < N} v(n) \]

**Theorem (H.–Krenn–Lechner 2024)**

Let \( x \) be a \( q \)-regular sequence with matrices \( (A_r)_{0 \leq r < q} \). Set \( C := \sum_{0 \leq r < q} A_r \). Assume that \( C \) has an eigenvalue \( \neq 0 \).

Then there is a non-negative integer \( k \) such that \( \Sigma^k x \) admits a “good asymptotic expansion”.

\[ \Sigma v(qN + r) = \sum_{0 \leq n < N} \sum_{0 \leq r' < q} v(qn + r') + \sum_{0 \leq r' < r} v(qN + r') \]

\[ = \sum_{0 \leq n < N} \sum_{0 \leq r' < q} A_{r'} v(n) + \sum_{0 \leq r' < r} A_{r'} v(N) \]
Iterated Summatory Function

$$\Sigma \nu(N) := \sum_{0 \leq n < N} \nu(n)$$

**Theorem (H.–Krenn–Lechner 2024)**

Let $x$ be a $q$-regular sequence with matrices $(A_r)_{0 \leq r < q}$. Set $C := \sum_{0 \leq r < q} A_r$. Assume that $C$ has an eigenvalue $\neq 0$.

Then there is a non-negative integer $k$ such that $\Sigma^k x$ admits a “good asymptotic expansion”.

$$\Sigma \nu(qN + r) = \sum_{0 \leq n < N} \sum_{0 \leq r' < q} A_{r'} \nu(n) + \sum_{0 \leq r' < r} A_{r'} \nu(N)$$

$$= \left( \sum_{0 \leq r' < q} A_{r'} \right) \sum_{0 \leq n < N} \nu(n) + \left( \sum_{0 \leq r' < r} A_{r'} \right) \nu(N)$$
Iterated Summatory Function

\[ \Sigma v(N) := \sum_{0 \leq n < N} v(n) \]

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Let \( x \) be a \( q \)-regular sequence with matrices \((A_r)_{0 \leq r < q}\). Set \( C := \sum_{0 \leq r < q} A_r \). Assume that \( C \) has an eigenvalue \( \neq 0 \).

Then there is a non-negative integer \( k \) such that \( \Sigma^k x \) admits a “good asymptotic expansion”.

\[
\Sigma v(qN + r) = \left( \sum_{0 \leq r' < q} A_{r'} \right) \sum_{0 \leq n < N} v(n) + \left( \sum_{0 \leq r' < r} A_{r'} \right) v(N)
\]

\[
= C \Sigma v(N) + \left( \sum_{0 \leq r' < r} A_{r'} \right) v(N)
\]
Iterated Summatory Function

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**Theorem (H.–Krenn–Lechner 2024)**

Let \( x \) be a \( q \)-regular sequence with matrices \( (A_r)_{0 \leq r < q} \). Set \( C := \sum_{0 \leq r < q} A_r \). Assume that \( C \) has an eigenvalue \( \neq 0 \).

Then there is a non-negative integer \( k \) such that \( \Sigma^k x \) admits a “good asymptotic expansion”.

\[
\Sigma v(qN + r) = \left( \sum_{0 \leq r' < q} A_{r'} \right) \sum_{0 \leq n < N} v(n) + \left( \sum_{0 \leq r' < r} A_{r'} \right) v(N)
\]

\[
= C \Sigma v(N) + \left( \sum_{0 \leq r' < r} A_{r'} \right) v(N)
\]

Repeating: asymptotic domination by \( D^k C \) (if \( C \) has an eigenvalue \( \neq 0 \)).
Divide-and-Conquer Sequences

Divide-and-conquer sequence:

\[ x(n) = \alpha x\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \beta x\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n) \]

for \( n \geq 2 \) (\( \alpha, \beta \) given positive constants, \( g \) given function ("toll function"), \( x(1) \) given).
Divide-and-Conquer Sequences

Divide-and-conquer sequence:

\[ x(n) = \alpha x\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \beta x\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n) \]

for \( n \geq 2 \) (\( \alpha, \beta \) given positive constants, \( g \) given function (“toll function“), \( x(1) \) given).

Theorem (Hwang–Janson–Tsai 2023)

Assume that there is an \( \varepsilon > 0 \) such that \( g(n) = O(n^{\log_2(\alpha+\beta)-\varepsilon}) \).

Then

\[ x(n) = n^{\log_2(\alpha+\beta)} \Phi(\{\log_2 n\}) + O(n^{\log_2(\alpha+\beta)-\varepsilon}) \]

for \( n \to \infty \) where \( \Phi \) is a continuous, 1-periodic function.
Divide-and-Conquer Sequences

Divide-and-conquer sequence:

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Theorem (Hwang–Janson–Tsai 2023)

Assume that there is an \( \varepsilon > 0 \) such that \( g(n) = O(n^{\log_2(\alpha+\beta)-\varepsilon}) \).

Then

\[ x(n) = n^{\log_2(\alpha+\beta)} \Phi(\{\log_2 n\}) + O(n^{\log_2(\alpha+\beta)-\varepsilon}) \]

for \( n \to \infty \) where \( \Phi \) is a continuous, 1-periodic function.

Question: Relation to “our“ result?
Divide-and-Conquer and Regular Sequences

\[ x(n) = \alpha x \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + \beta x \left( \left\lceil \frac{n}{2} \right\rceil \right) + g(n) \]
Divide-and-Conquer and Regular Sequences

\[ x(n) = \alpha x\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + \beta x\left(\left\lceil \frac{n}{2} \right\rceil \right) + g(n) \]

\[ \iff \]

\[ x(2n) = (\alpha + \beta)x(n) + g(2n) \]
\[ x(2n + 1) = \alpha x(n) + \beta x(n + 1) + g(2n + 1) \]
Divide-and-Conquer and Regular Sequences

\[ x(n) = \alpha x\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \beta x\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n) \]

\[ \iff \]

\[ x(2n) = (\alpha + \beta)x(n) + g(2n) \]
\[ x(2n + 1) = \alpha x(n) + \beta x(n + 1) + g(2n + 1) \]

\[ \iff \text{(with } v(n) = (x(n), x(n + 1))^\top) \]

\[ v(2n) = \begin{pmatrix} \alpha + \beta & 0 \\ \alpha & \beta \end{pmatrix} v(n) + \begin{pmatrix} g(2n) \\ g(2n + 1) \end{pmatrix} \]
\[ v(2n + 1) = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha + \beta \end{pmatrix} v(n) + \begin{pmatrix} g(2n + 1) \\ g(2n + 2) \end{pmatrix} \]
Divide-and-Conquer and Regular Sequences

\[ x(n) = \alpha x\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + \beta x\left(\left\lceil \frac{n}{2} \right\rceil \right) + g(n) \]

\[ \iff \]

\[ x(2n) = (\alpha + \beta)x(n) + g(2n) \]
\[ x(2n + 1) = \alpha x(n) + \beta x(n + 1) + g(2n + 1) \]

\[ \iff \]

\[ \text{(with } v(n) = (x(n), x(n + 1))^\top \text{)} \]

\[ v(2n) = \begin{pmatrix} \alpha + \beta & 0 \\ \alpha & \beta \end{pmatrix} v(n) + \begin{pmatrix} g(2n) \\ g(2n + 1) \end{pmatrix} \]
\[ v(2n + 1) = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha + \beta \end{pmatrix} v(n) + \begin{pmatrix} g(2n + 1) \\ g(2n + 2) \end{pmatrix} \]

If \( g \) is regular, then \( x \) is regular.
Summatory Function of the Forward Difference

\[ x(N) = x(0) + \sum_{0 \leq n < N} (x(n + 1) - x(n)) \]
Summatory Function of the Forward Difference

\[ x(N) = x(0) + \sum_{0 \leq n < N} (x(n + 1) - x(n)) \]

- Forward difference of regular sequence is regular
- Summatory function of regular sequence is regular
  
  . . . but can we say something in general?
Forward Difference of Divide-and-Conquer Sequence

Set $\Delta x(n) := x(n + 1) - x(n)$. 
Forward Difference of Divide-and-Conquer Sequence

Set $\Delta x(n) := x(n + 1) - x(n)$.
Consider divide-and-conquer sequence

\[ x(2n) = (\alpha + \beta)x(n) + g(2n) \]
\[ x(2n + 1) = \alpha x(n) + \beta x(n + 1) + g(2n + 1) \]
Forward Difference of Divide-and-Conquer Sequence

Set $\Delta x(n) := x(n + 1) - x(n)$.

Consider divide-and-conquer sequence

$$
\begin{align*}
x(2n) &= (\alpha + \beta)x(n) + g(2n) \\
x(2n + 1) &= \alpha x(n) + \beta x(n + 1) + g(2n + 1)
\end{align*}
$$

$$
\Rightarrow
\begin{align*}
\Delta x(2n) &= \beta \Delta x(n) + g(2n + 1) - g(2n) \\
\Delta x(2n + 1) &= \alpha \Delta x(n) + g(2n + 2) - g(2n + 1).
\end{align*}
$$
Set $\Delta x(n) := x(n + 1) - x(n)$.
Consider divide-and-conquer sequence

\[
x(2n) = (\alpha + \beta)x(n) + g(2n)
\]
\[
x(2n + 1) = \alpha x(n) + \beta x(n + 1) + g(2n + 1)
\]

$\Rightarrow$

\[
\Delta x(2n) = \beta \Delta x(n) + g(2n + 1) - g(2n)
\]
\[
\Delta x(2n + 1) = \alpha \Delta x(n) + g(2n + 2) - g(2n + 1).
\]

Dimension 1 (plus dimension of linear representation of $g$): particulary simple.
**Divide-and-Conquer: Result**

<table>
<thead>
<tr>
<th>Theorem (H.–Krenn–Lechner 2024)</th>
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<tbody>
<tr>
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Theorem (H.–Krenn–Lechner 2024)

Let \( x \) be a divide-and-conquer sequence with polynomial toll function of degree \( k \geq 1 \). Then (for 1-periodic continuous functions \( \Phi \) and \( \Psi \) and \( n \to \infty \)):

- **Case 1a.** If \( \alpha + \beta > 2^k \) and \( 2^k > \max\{\alpha, \beta\} \), then

\[
x(n) = n^{\log_2(\alpha + \beta)} \Phi(\{\log_2 n\}) + n^k \Psi(\{\log_2 n\}) + O(n^{\log_2 \max\{\alpha, \beta\}}).
\]
Theorem (H.–Krenn–Lechner 2024)

Let $x$ be a divide-and-conquer sequence with polynomial toll function of degree $k \geq 1$. Then (for 1-periodic continuous functions $\Phi$ and $\Psi$ and $n \to \infty$):

- **Case 1b.** If $\alpha + \beta > 2^k$ and $\max\{\alpha, \beta\} \geq 2^k$, then

$$x(n) = n^{\log_2(\alpha+\beta)} \Phi(\{\log_2 n\})$$

$$+ \mathcal{O}(n^{\log_2 \max\{\alpha, \beta\}} (\log n)^{\max\{\alpha, \beta\}=2^k}).$$
Theorem (H.–Krenn–Lechner 2024)

Let \( x \) be a divide-and-conquer sequence with polynomial toll function of degree \( k \geq 1 \). Then (for 1-periodic continuous functions \( \Phi \) and \( \Psi \) and \( n \to \infty \)):

- **Case 2.** If \( \alpha + \beta = 2^k \), then

\[
x(n) = n^k(\log n)\Phi(\{\log_2 n\}) + n^k\Psi(\{\log_2 n\}) + O\left(n^{\log_2 \max\{\alpha, \beta\} + [\alpha=\beta] \varepsilon}\right)
\]

for any \( \varepsilon > 0 \).
Theorem (H.–Krenn–Lechner 2024)

Let \( x \) be a divide-and-conquer sequence with polynomial toll function of degree \( k \geq 1 \). Then (for 1-periodic continuous functions \( \Phi \) and \( \Psi \) and \( n \to \infty \)):

- **Case 3.** If \( 2^k > \alpha + \beta > 2^{k-1} \), then

\[
x(n) = n^k \Phi\left(\{\log_2 n\}\right) + n^{\log_2(\alpha+\beta)} \Psi\left(\{\log_2 n\}\right) + O\left(n^{\log_2 \max\{\alpha,\beta,2^{k-1}\} + \left[\max\{\alpha,\beta\} = 2^{k-1}\right] \varepsilon \right)
\times (\log n)^{\left[\max\{\alpha,\beta\} < 2^{k-1}\right]}
\]

for any \( \varepsilon > 0 \).
Divide-and-Conquer: Result

Theorem (H.–Krenn–Lechner 2024)

Let $x$ be a divide-and-conquer sequence with polynomial toll function of degree $k \geq 1$. Then (for 1-periodic continuous functions $\Phi$ and $\Psi$ and $n \to \infty$):

- Case 4. If $2^{k-1} \geq \alpha + \beta$, then

$$x(n) = n^k \Phi(\{\log_2 n\}) + O(n^{k-1}(\log n)^E),$$

where

$$E := 1 + [\alpha + \beta = 2^{k-1}][[k \geq 2 \text{ and } c_{k-1} \neq 0]$$

$$+ [k = 1 \text{ and } d_0 + d_1 \neq 0]]$$

with

$$d_0 := (1 - \beta)x(1) - g(1) + g(0), \quad d_1 := g(1) - (1 - \beta)x(1).$$
Min-Max: Fourier Coefficients

Computing Fourier coefficients using the general result ...