Weak Separation Property & Finite Type Condition

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Definition

Define

$$\Sigma_a = \{\{c_i\}_{i=1}^{\infty} : c_i \in \{0, a, 6\}\}.$$

There is a natural projection $\pi : \Sigma_a \to \mathbb{R}$ given by

$$\pi(\{c_i\}) = \sum_{i=1}^{\infty} \frac{c_i}{7^i}.$$

Define $K_a = \pi(\Sigma_a)$. For $x \in K_a$ we define the set of addresses of x as

$$\pi^{-1}(x) = \{\{c_i\} : \pi(\{c_i\}) = x\}.$$

Fact

All $x \in K_a$ have at least one address. Depending on a and x, it is possible for x to have many more.

Definition (Alternate Definition)

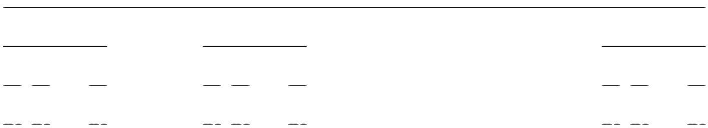
Consider $S_i(x) = \frac{x+i}{7}$. Then K_a is the unique non-empty compact set such that

$$K_a = S_0(K_a) \cup S_a(K_a) \cup S_6(K_a)$$

Here we call $\{S_0, S_a, S_6\}$ an iterated function system (IFS), and K_a the attractor for the IFS.

For this talk, we will always assume equicontractive and $\mathrm{hull}(K) = [0, 1]$.

Let $a = 2$.



We see
$S_0(K_2) \cap S_2(K_2) = S_0(K_2) \cap S_6(K_2) = S_2(K_2) \cap S_6(K_2) = \emptyset$.
This allows us to claim

Fact

For all $x \in K_2$, x has a unique addresss

This works for all $1 < a < 5$.

This is part of a more general property.

Definition

We say an IFS $\{F_i\}$ satisfies the strong separation property (SSP) if $F_i(K) \cap F_j(K) = \emptyset$ for all $i \neq j$.

Fact

For an IFS satisfying the SSP, every address is unique. The Hausdorff dimension is easily computed. For example $\dim_H(K_2) = \log(3)/\log(7) \approx 0.5645$.

Let $a = 1$.

We see $S_0(K_1) \cap S_1(K_1) = 1/7$. In particular

$$\pi^{-1}(1/7) = \{\{0, 6, 6, 6, \dots\}, \{1, 0, 0, 0, \dots\}\}$$

Fact

For all $\sigma \in \{0, 1, 6\}^*$ we have $\pi(\sigma 0666 \dots) = \pi(\sigma 1000 \dots)$. All other points have a unique address.

This is part of a more general property.

Definition

We say an IFS $\{F_i\}$ satisfies the open set condition (OSC) if there exists an open set V such that $F_i(V) \cap F_j(V) = \emptyset$ for all $i \neq j$ and $F_(V) \subset V$ for all i..

Definition

We say the IFS $\{F_i\}$ satisfies the convex open set condition ($\mathrm{OSC_{co}}$) if $V = (0, 1)$.

Fact

For an IFS satisfying the OSC, almost every address is unique. The Hausdorff dimension is easily computed. For example $\dim_H(K_1) = \log(3)/\log(7) \approx 0.5645$.

Let $a = 6/7$.

We see $S_0 \circ S_6 = S_a \circ S_0$. This means that if x has address $c_1 c_2 c_3 \ldots$ we can replace any occurance of 06 with $a0$ and vice-a-versa. For example

- $x = \pi(a00000\ldots)$ has two addresses,
- $x = \pi(066666\ldots)$ has countably many addresses and
- $x = \pi(060606\ldots)$ has uncountably many addresses.

Let $\sigma = c_1 c_2 \dots c_m \in \{0, a, 6\}^*$. We define $S_\sigma = S_{c_1} \circ S_{c_2} \circ \cdots \circ S_{c_m}$.

Definition

We say an IFS $\{F_i\}$ satisfies the weak separation property (WSP) if there exists a $c > 0$ such that for all σ, τ with $|\sigma| = |\tau|$ either $S_\sigma = S_\tau$ or $\frac{S_\sigma(0) - S_\tau(0)}{|S_\sigma([0,1])|} \geq c$.

Definition

Let σ, τ be finite words with $|\sigma| = |\tau|$. For an open set V we say that σ and τ are neighbours if $S_\sigma(V) \cap S_\tau(V) \neq \emptyset$.

Definition

We say an IFS $\{F_i\}$ satisfes the finite type condition (FTC) if there exists a V such that there are only finitelly many neighbourhood sets

$$\{S_\sigma^{-1} \circ S_\tau : \sigma \text{ is a neighbour of } \tau\}$$

Definition

We say an IFS $\{F_i\}$ satisifes the convex finite type condition (FTC_{co}) if $V = (0, 1)$.

$$\begin{array}{ccccccc} & & \mathrm{OSC_{co}} & \Rightarrow & \mathrm{FTC_{co}} & & \\ & & \Downarrow & & \Downarrow & \searrow & \\ \mathrm{SSP} & \Rightarrow & \mathrm{OSC} & \Rightarrow & \mathrm{FTC} & & \mathrm{WSP} \end{array}$$

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Take $a = 1/2$ for K_a

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Nguyen 2002

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- Let $S_{\sigma_n}(0) < S_{\tau_n}(0)$. Choose a, c_{n+1}, c'_{n+1} so that $\sigma_{n+1} = \sigma_n c_{n+1}$ and $\tau_{n+1} = \tau_n c'_{n+1}$ are neighbours, and either $S_{\sigma_{n+1}}(0) < S_{\tau_{n+1}}(0)$ or $S_{\tau_{n+1}}(0) < S_{\sigma_{n+1}}(0)$.

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- Similarly if $S_{\sigma_n}(0) > S_{\tau_n}(0)$.
- This gives us a sequences $0c_2c_3\ldots$ and $ac'_2c'_3\ldots$ such that for each n, $\sigma_n = 0c_2c_3\ldots c_n$ and $\tau_n = ac'_2c'_3\ldots c'_n$ are neighbours.

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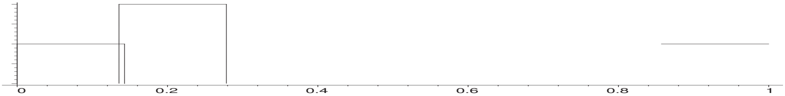
- If we choose between these options in an aperiodic way, then we have an example that have an infinite number of neighbourhood types.



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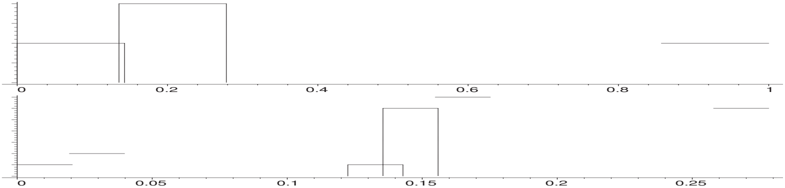
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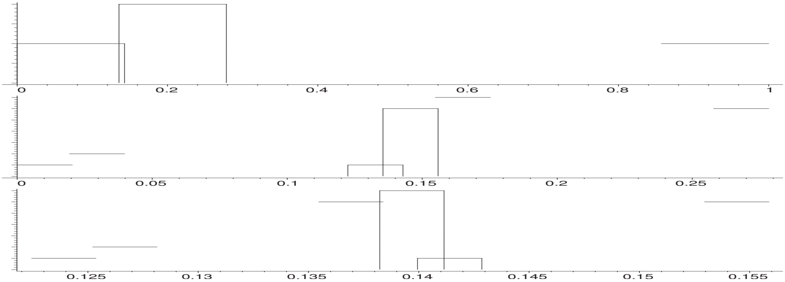
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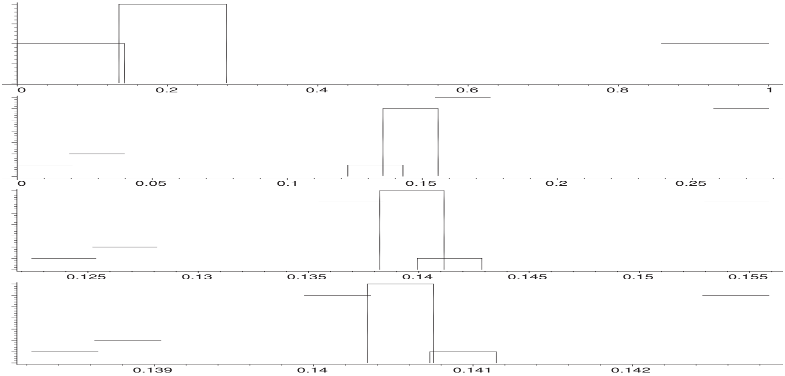

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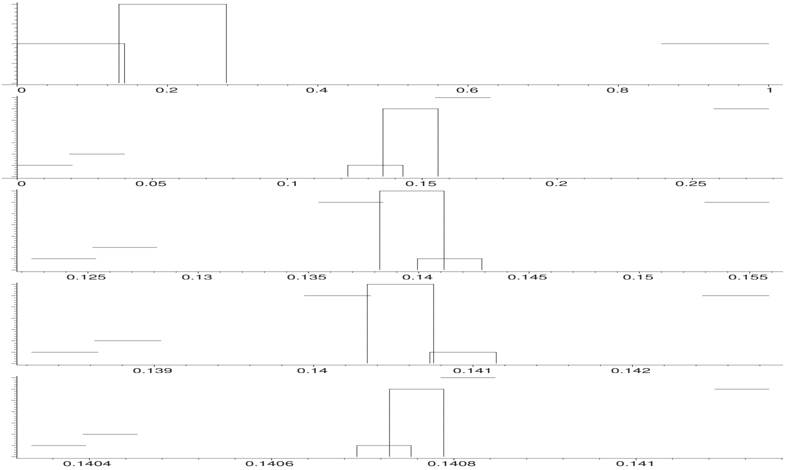
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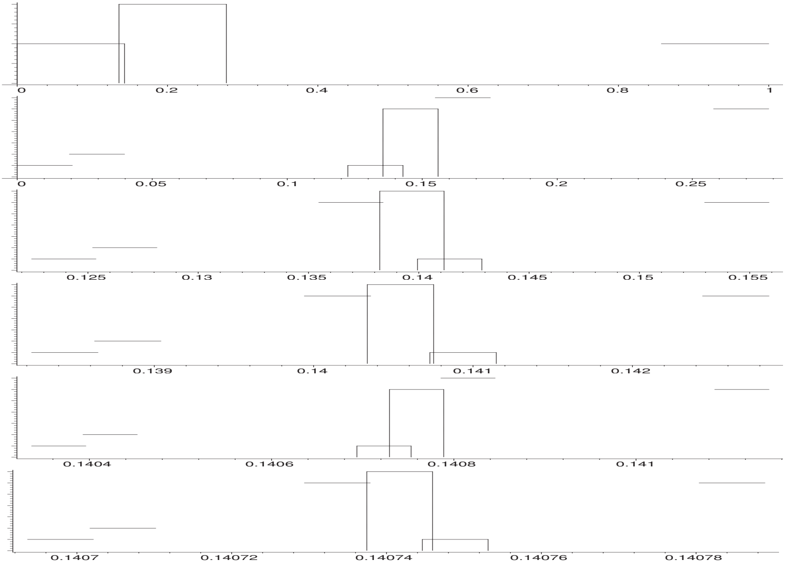
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Fact

Let $V = \cup_\sigma S_\sigma((3/7, 4/7))$. *Then* $\{S_0, S_a, S_6\}$ *satisfy OSC with this* V.

Fact

It is possible to modify this example so that it does not satisfy OSC, nor FTC_{co}*, but does satisfy FTC.*

Fact

For all $\sigma \in \{0, a, 6\}^*$ *we have* $\pi(\sigma c_1 c_2 \dots) = \pi(\sigma c_1' c_2' \dots)$. *That is, we have a countable number of points with two non-periodic addresses, and all other points have unique addresses.*

Thank you