Notions of normality for dynamically generated basic sequences

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Basic Sequences, normality, and distribution normality

A sequence $Q = (q_n)_{n=1}^{\infty} \in \mathbb{N}_{\geq 2}^{\mathbb{N}}$ is called a **basic sequence**. We let $\tilde{Q} = \prod_{n=1}^{\infty}[0, q_n - 1]$ denote the space of Q-Cantor series of the elements of $[0,1)$ under the correspondence given by

$$x = \sum_{n=1}^{\infty} \frac{a_n}{q_1 q_2 \cdots q_n}, \tag{1}$$

where we assume without loss of generality that $a_n \neq q_n - 1$ infinitely often. We may abbreviate this correspondence by writing $x = 0.a_1 a_2 \cdots a_n \cdots_Q$, and we let $\phi_Q : \tilde{Q} \to [0,1)$ be the map given by $\phi_Q(a_n)_{n=1}^{\infty} = x$. We say that $(a_n)_{n=1}^{\infty} \in \tilde{Q}$ is **Q-distribution normal** if the sequence $(q_n q_{n-1} \cdots q_1 \phi_Q(a_m)_{m=1}^{\infty})_{n=1}^{\infty}$ is uniformly distributed mod 1, and we let $\mathcal{DN}(Q)$ denote the collection of Q-distribution normal sequences. The sequence $(a_n)_{n=1}^{\infty}$ is **Q-normal** if all blocks of digits occur with the expected frequency (for which we omit the precise definition), and we denote the collection of Q-normal sequences by $\mathcal{N}(Q)$.

Dynamically generated basic sequences

A basic sequence $Q = (q_n)_{n=1}^{\infty}$ is **dynamically generated** if there exists a separable metric space X, a probability measure on $(X, \mathscr{B})$ with full topological support where $\mathscr{B}$ being the Borel σ-algebra, a continuous μ-preserving transformation $T : X \to X$, a generic point $x \in X$ for (T, μ), and a continuous $f : X \to \mathbb{N}_{\geq 2}$ for which $q_n = f(T^n x)$. If the system $(X, \mathscr{B}, \mu, T)$ has zero entropy, then Q is **deterministic**. We also have the following characterization of dynamically generated basic sequences.

Theorem 1: A basic sequence $Q = (q_n)_{n=1}^{\infty}$ is dynamically generated if and only if:

(i) For any $w := [w_1, w_2, \cdots, w_k] \in \mathbb{N}_{\geq 2}^k$ the following limit exists

$$d(w) := \lim_{N\to\infty} \frac{1}{N} \left|\{1 \leq n \leq N \mid q_{n+i} = w_i \ \forall\, 0 \leq i < k\}\right|. \tag{2}$$

(ii) If $d(w) = 0$, then $w \neq [q_n, q_{n+1}, \cdots, q_{n+k-1}]$ for any $n \in \mathbb{N}$.

(iii) For any $k \in \mathbb{N}$, we have $\sum_{|w|=k} d(w) = 1$.

(iv) Q is deterministic if and only if it has 'subexponential measurable word complexity'.

Examples of basic sequences

1. If $b \in \mathbb{N}_{\geq 2}$ and $Q = (b)_{n=1}^{\infty}$, then $\mathcal{N}(Q) = \mathcal{DN}(Q)$, and $\mathcal{N}(Q)$ is precisely the set of numbers that are normal base-b.
2. If $Q = (q_n)_{n=1}^{\infty}$ with $q_{2n} = 2$ and $q_{2n-1} = 3$, then $\mathcal{N}(Q) = \mathcal{DN}(Q)$, and $\mathcal{N}(Q)$ is precisely the set of numbers that are normal base-6. Furthermore, we see that Q is a dynamically generated basic sequence. In fact, any periodic Q is deterministic and dynamically generated.
3. Let $Q = (q_n)_{n=1}^{\infty}$ be such that $\lim_{n\to\infty} q_n = \infty$ and $\sum_{n=1}^{\infty} \frac{1}{q_n} = \infty$. Then, $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ and $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ are $D_2(\Pi_3^0)$-complete. A specific example of such a Q is given by $q_n = n + 1$.
4. One representation of the Thue-Morse sequence $(d_n)_{n=1}^{\infty}$ is given by $d_n = (-1)^{c_n}$, where c_n is the number of 1s appearing in the base 2 expansion of n. For any $a, b \geq 2$, the sequence $Q = Q(a,b) = (q_n(a,b))_{n=1}^{\infty}$ given by $q_n(a,b) = a$ if $d_n = 1$ and $q_n(a,b) = b$ if $d_n = -1$ is a deterministic dynamically generated basic sequence.
5. Let $(d_n)_{n=1}^{\infty}$ be the digits of the base 10 expansion of the Champernowne number. The sequence $Q_C = (q_n)_{n=1}^{\infty}$ given by $q_n = d_n + 2$ is a non-deterministic dynamically generated basic sequence.
6. Let $(d_n)_{n=1}^{\infty}$ be the digits of the continued fraction expansion of the Adler-Keane-Smorodinsky number. The sequence $Q = (q_n)_{n=1}^{\infty}$ given by $q_n = d_n + 1$ is a non-deterministic dynamically generated basic sequence. We note that $\limsup_{n\to\infty} q_n = \infty$.

Uniform normality and uniform distribution normality

Let $Q = (q_n)_{n=1}^{\infty}$ be a dynamically generated basic sequence. $x \in [0,1]$ is **uniformly Q-normal** if the pairs of bases and digits that appear in the base Q expansion of x appear with the correct frequency. To be more precise, let us fix a block $D = (d_1, \cdots, d_K)$ of potential digits and a block $B = (b_1, \cdots, b_K)$ of potential bases. Let $S_B = \{j \in \mathbb{N} \mid (q_j, q_{j+1}, \cdots, q_{j+K-1}) = (b_1, b_2, \cdots, b_K)\}$, and define

$$Q_n(D, B) = \sum_{j \in S_B \cap [1,n]} \frac{\mathcal{I}_{Q,j}(D)}{b_1 b_2 \cdots b_K}, \text{ and} \tag{3}$$

$$N_n^D(Q, z, B) = \#\{i \in S_B \cap [1,n] :\ d_j = E_{i+j}(z),\ \forall\, 1 \leq j \leq k\}. \tag{4}$$

We say that $z = 0.E_1 E_2 \cdots E_n \cdots_Q$ is **uniformly Q-normal** if for all blocks D and B with $D < B$ and $\lim_{n\to\infty} Q_n(D,B) = \infty$ we have

$$\lim_{n\to\infty} \frac{N_n^D(Q, z, B)}{Q_n(D, B)} = 1, \tag{5}$$

and we denote the collection of such sequences by $\mathcal{UN}(Q)$. Now let $(X, \mathscr{B}, \mu, T)$ be a system corresponding to Q and $f \in C(X)$ a function corresponding to Q. We say that $(E_n)_{n=1}^{\infty}$ is **uniformly Q-distribution normal** if (x, y) has a uniformly distributed orbit in the skew product system $(X \times [0,1], \mathscr{B} \times \mathscr{L}, \mu \times m, T \rtimes M)$, where $M_n z = nz \pmod 1$, $T \rtimes M(x, z) = (Tx, M_{f(x)} z)$, $\mathscr{L}$ is the Lebesgue σ-algebra on $[0,1]$ and m is the Lebesgue measure. We denote the collection of such sequences by $\mathcal{UDN}(Q)$. It is worth noting that a priori, the definition of uniformly Q-distribution normal sequences depends on the system corresponding to Q and the function corresponding to Q. Our next result shows that this is not the case.

Theorem 2: If $Q = (q_n)_{n=1}^{\infty}$ is a dynamically generated basic sequence, then $\mathcal{UN}(Q) = \mathcal{UDN}(Q)$.

Results for deterministic dynamically generated basic sequences

Theorem 3: Let $g \in \mathbb{N}_{\geq 2}$ and let $Q = (q_n)_{n=1}^{\infty} \in \left(\{g^n\}_{n=1}^{\infty}\right)^{\mathbb{N}}$ be a deterministic dynamically generated basic sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log(q_n) < \infty. \tag{6}$$

For $y \in [0,1]$ the following are equivalent:

(i) y is normal base g. (ii) y is uniformly Q-normal. (iii) y is Q-distribution normal (iv) y is Q-normal.

Theorem 4: If Q is a deterministic dynamically generated basic sequence, then $\mathcal{N}(Q) = \mathcal{DN}(Q) = \mathcal{UN}(Q)$.

Theorem 5 (Hot spot): Let $Q = (q_n)_{n=1}^{\infty}$ be a deterministic dynamically generated basic sequence. Let $x \in [0,1)$ be such that for any $\sigma \in (0,1)$, there exists a subset $\mathcal{N}_\sigma \subseteq \mathbb{N}$ of upper density at most σ, and a constant $C > 0$ (not dependent on σ) such that the following holds: for every $0 \leq a < b \leq 1$ we have

$$\limsup_{N\to\infty} \frac{\nu_{(a,b)}(x, N; \mathcal{N}_\sigma)}{N} < C(b-a)^{\sigma}, \tag{7}$$

where $\nu_{(a,b)}(x, N; \mathcal{N}_\sigma) = \#\{n \in [0, N-1] \setminus \mathcal{N}_\sigma :\ \prod_{i=1}^{n} q_i x \in (a,b)\}$. Then $x \in \mathcal{DN}(Q)$.

Examples involving dynamically generated basic sequences

Example 1: Let Q_C be as in Example 5, which is generated by a Bernoulli system. We have

(i) $\dim_H(\mathcal{N}(Q_C) \setminus \mathcal{DN}(Q_C)) = \dim_H(\mathcal{DN}(Q_C) \setminus \mathcal{N}(Q_C)) = 1$.

(ii) $\mathcal{DN}(Q_C) \setminus \mathcal{N}(Q_C)$ and $\mathcal{N}(Q_C) \setminus \mathcal{DN}(Q_C)$ are both $D_2(\Pi_3^0)$-complete.

(iii) There exists $x \in \mathcal{N}(Q_C)$ such that for $M_n := \prod_{j=1}^{n} q_j$, the sequence $\{\frac{1}{N}\sum_{n=1}^{N} \delta_{M_N x}\}_{N=1}^{\infty}$ converges in the weak* topology to a measure μ satisfying

$$\mu([a,b]) = \begin{cases} \frac{5}{6}(b-a) & \text{if } 0 \leq a < b \leq \frac{1}{2} \\ \frac{7}{6}(b-a) & \text{if } \frac{1}{2} \leq a < b \leq 1. \end{cases} \tag{8}$$

In particular, Q_C does not admit a Hot Spot Theorem.

Example 2: Let $g \in \mathbb{N}_{\geq 2}$ and let $Q = (q_n)_{n=1}^{\infty} \in \left(\{g^n\}_{n=1}^{\infty}\right)^{\mathbb{N}}$ be a dynamically generated basic sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log(q_n) = \infty. \tag{9}$$

(i) There exists $y \in [0,1] \setminus (\mathcal{N}(Q) \cup \mathcal{DN}(Q))$ that is normal base g.

(ii) There exists $y \in \mathcal{UN}(Q)$ that is not normal base g.

Questions and conjectures for future work

Question 1: Is $\left\{Q \in \mathbb{N}_{\geq 2}^{\mathbb{N}} \mid \mathcal{N}(Q) = \mathcal{DN}(Q)\right\}$ a Borel set?

Question 2: Are there selection rules for deterministic dynamically generated basic sequences Q? If Q is not deterministic, then this question can be asked separately for $\mathcal{N}(Q), \mathcal{DN}(Q)$, and $\mathcal{UN}(Q)$.

Question 3: If Q is a non-deterministic dynamically generated basic sequence, do we have $\mathcal{UN}(Q) = \mathcal{N}(Q) \cap \mathcal{DN}(Q)$? If not, what can be said about the Hausdorff dimension and descriptive complexity of $(\mathcal{N}(Q) \cap \mathcal{DN}(Q)) \setminus \mathcal{UN}(Q)$?

Question 4: Given a deterministic dynamically generated basic sequence Q, what is the set of $y \in [0,1]$ for which $y + \mathcal{N}(Q) \subseteq \mathcal{N}(Q)$? If Q is non-deterministic, then this question can be asked separately for $\mathcal{N}(Q), \mathcal{DN}(Q)$, and $\mathcal{UN}(Q)$.

Question 5: Given a deterministic dynamically generated basic sequence Q, what is the set of $q \in \mathbb{Q}$ for which $q\mathcal{N}(Q) \subseteq \mathcal{N}(Q)$? If Q is non-deterministic, then this question can be asked separately for $\mathcal{N}(Q), \mathcal{DN}(Q)$, and $\mathcal{UN}(Q)$. We observe that $n\mathcal{DN}(Q) \subseteq \mathcal{DN}(Q)$ for any Q and any $n \in \mathbb{N}$.

Conjecture 1: If Q is a non-deterministic dynamically generated basic sequence, then

(i) $\dim_H(\mathcal{N}(Q) \setminus \mathcal{DN}(Q)) = \dim_H(\mathcal{DN}(Q) \setminus \mathcal{N}(Q)) = 1$

(ii) $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ and $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ are both $D_2(\Pi_3^0)$-complete.

Conjecture 2: If Q is a non-deterministic dynamically generated basic sequence, then there exists $x \in \mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ and $C > 0$ for which

$$\limsup_{N\to\infty} \frac{1}{N} \left|\{1 \leq n \leq N \mid \prod_{i=1}^{n} q_i x \in (a,b)\}\right| \leq C(b-a), \tag{10}$$

for all $0 \leq a < b \leq 1$.