Pitch for notions of normality for dynamically generated basic sequences

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Theory of normality base b

Let $\mathcal{N}_b \subseteq [0,1]$ denote the set of numbers that are normal base b.

1. For $x \in [0,1]$ we have $x \in \mathcal{N}_b$ if and only if $(b^n x)_{n=1}^{\infty}$ is uniformly distributed.
2. Rauzy [Rau76] characterized the $y \in [0,1]$ for which $y + \mathcal{N}_b \subseteq \mathcal{N}_b$ using an entropy-like condition called noise.
3. The work of Kamae and Weiss [KW75, Kam73] characterizes those $(a_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ for which $x = 0.x_1 x_2 \cdots x_n \cdots \in \mathcal{N}_b$ implies $x' = 0.x_{a_1} x_{a_2} \cdots x_{a_n} \cdots \in \mathcal{N}_b$.
4. The Hot Spot Theorem of Pyatetski-Shapiro [Pv57] says that if $x \in [0,1]$ is such that $(b^n x)_{n=1}^{\infty}$ is 'almost' uniformly distributed, then $x \in \mathcal{N}_b$.

Theory of normality for continued fractions

A number $x \in [0,1]$ is **continued fraction (c.f.) normal** if $(T^n x)_{n=1}^{\infty}$ is uniformly distributed with respect to the Gauss measure, where $T : [0,1] \to [0,1]$ is the Gauss map.

1. $x \in [0,1]$ is c.f. normal if and only if for any $\ell \in \mathbb{N}$ and any $w \in \mathbb{N}^{\ell}$, the word w appears in the c.f. expansion of x with the 'correct' frequency.
2. Vandehey [Van17] showed that if that $a,b,c,d \in \mathbb{Z}$ are such that $ad - bc \neq 0$, and x is c.f. normal, then so is $\frac{ax+b}{cx+d}$.
3. Vandehey and Heersink [HV16] showed that if $x = [x_1, x_2, \cdots, x_n, \cdots] \in [0,1]$ is continued fraction normal, and $a, b \in \mathbb{N}$ with $b \geq 2$, then $[x_a, x_{a+b}, \cdots, x_{a+nb}, \cdots]$ is **NOT** continued fraction normal.
4. Shkredov and Moshchevitin [MS03, AM20, Shk10] proved a Hot Spot Theorem for continued fractions.

Theory of normality for general Cantor series

Let $Q \in (\mathbb{N}_{\geq 2})^{\mathbb{N}}$ be a basic sequence. $x \in [0,1]$ is Q-**distribution normal** if $(q_n \cdots q_1 x)_{n=1}^{\infty}$ is uniformly distributed, and x is Q-**normal** if all blocks of potential digits in the base Q expansion of x appear with the "correct" frequency. Let $\mathcal{DN}(Q)$ and $\mathcal{N}(Q)$ be the set of Q-distribution normal and Q-normal numbers.

Theorem (Airey, Jackson, and Mance [AJM22])

If $Q = (q_n)_{n=1}^{\infty}$ is such that $\lim_{n\to\infty} q_n = \infty$ and $\sum_{n=1}^{\infty} q_n^{-1} = \infty$, then $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ and $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ are $D_2(\Pi_3^0)$-complete.

This shows that for many Cantor series we are lacking the necessary connection between combinatorics and dynamics that is used to develop a rich theory of normality. This is because we can only associate a general basic sequence to a *non-autonomous* dynamical system. Is there a reasonable class of Cantor series for which we have $\mathcal{DN}(Q) = \mathcal{N}(Q)$?

Dynamically generated basic sequences

A basic sequence $Q = (q_n)_{n=1}^{\infty}$ is **dynamically generated** if there exists a "nice" measure preserving system $\mathcal{X} := (X, \mathscr{B}, \mu, T)$, a function $f : X \to \mathbb{N}_{\geq 2}$, and a generic point $x \in X$ for which $q_n = f(T^n x)$. Normality and distribution normality for the basic sequence Q is naturally studied through a dynamical system $\mathcal{X}'$ that is a skew-product over $\mathcal{X}$. This allows for the new notions of **uniform normality** $\mathcal{UN}(Q)$ and **uniform distribution normality** $\mathcal{UDN}(Q)$, as well as

1. The relation $\mathcal{UN}(Q) = \mathcal{UDN}(Q)$.
2. The relation $\mathcal{N}(Q) = \mathcal{DN}(Q) = \mathcal{UN}(Q)$ when $\mathcal{X}$ has zero entropy.
3. A Hot Spot Theorem for $\mathcal{DN}(Q)$ when $\mathcal{X}$ has zero entropy.
4. Many interesting counter-examples when $\mathcal{X}$ is a Bernoulli system.

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