Numbers expressible by quotients or differences of two Pisot numbers

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Definitions

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Pisot and Salem numbers appear in various contexts of number theory, harmonic analysis, dynamical systems, tillings, etc.

Pisot numbers

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proved that its smallest element is the real root $\theta = 1.32471\ldots$ of $x^3 - x - 1$.

Earlier results on Pisot numbers

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Theorem 2

Every positive algebraic number is a quotient of two Mahler measures.

Mahler measure

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$$M(\alpha) \geq \alpha$$

with equality if and only if α is a Salem or a Pisot number.

Every positive algebraic number is a quotient of Pisot numbers

Therefore, the following theorem, recently proved in

A. Dubickas, Every Salem number is a difference of two Pisot numbers, *Proc. Edinb. Math. Soc.*, **66** (2023), 862–867,

generalizes both Theorem 1 and Theorem 2:

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generalizes both Theorem 1 and Theorem 2:

Theorem 3

Every real positive algebraic number α of degree d is expressible as a quotient of two Pisot numbers of degree d from the field $\mathbb{Q}(\alpha)$.

Additive version

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Which numbers are expressible as differences of two Pisot numbers?

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More explicitly, I showed that

Theorem 5

For each Salem number α of degree $d \geq 4$ there exist infinitely many $n \in \mathbb{N}$ for which $\alpha^{2n-1} - \alpha^n + \alpha$ and $\alpha^{2n-1} - \alpha^n$ are both Pisot numbers of degree d. The smallest such n is at most $6^{d/2-1} + 1$.

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It is clear that each such α must be a real algebraic integer since so are β and γ.

As we will show below, not every real algebraic integer α is expressible as a difference of two Pisot numbers due to several restrictions on the conjugates of α.

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Our next theorem shows that, under the assumption that the representation (1) is possible, the Pisot numbers β and γ in it can always be chosen to be in the field $\mathbb{Q}(\alpha)$.

Theorem 6

If α is expressible as a difference of two Pisot numbers, then there exist Pisot numbers $\beta, \gamma \in \mathbb{Q}(\alpha)$ for which $\alpha = \beta - \gamma$.

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Theorem 6

If α is expressible as a difference of two Pisot numbers, then there exist Pisot numbers $\beta, \gamma \in \mathbb{Q}(\alpha)$ for which $\alpha = \beta - \gamma$.

This, and other results below, is in:

A. Dubickas, Numbers expressible as a difference of two Pisot numbers, *Acta Mathematica Hungarica*, **172** (2024), 346–358.

Degree 1

Let d be the degree of α over $\mathbb{Q}$. The situation is very simple for $d = 1$. Then, $\alpha \in \mathbb{Z}$ and we clearly have

$$\alpha = (m + \alpha) - m,$$

where m is a positive integer satisfying $m \geq \max(2, 2 - \alpha)$. Observe that $m + \alpha$ and m are both rational integers greater than or equal to 2, so both are Pisot numbers.

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where m is a positive integer satisfying $m \geq \max(2, 2 - \alpha)$. Observe that $m + \alpha$ and m are both rational integers greater than or equal to 2, so both are Pisot numbers. Of course, other representations are also possible. For example, $\alpha = 1$ is the difference of two quadratic Pisot numbers $2 + \sqrt{2}$ and $1 + \sqrt{2}$ that are outside the field $\mathbb{Q}(\alpha) = \mathbb{Q}$.

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In all that follows we, therefore, assume that $d \geq 2$. We now show that the conjugates of α expressible as a difference of two Pisot numbers as in (1) must all lie in the union of the open disc

$$D = \{z \in \mathbb{C} \ : \ |z| < 2\}$$

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$$D = \{z \in \mathbb{C} \ : \ |z| < 2\}$$

and the infinite strip

$$S = \{z \in \mathbb{C} \ : \ |\Im z| < 1\}.$$

Degree d (continuation)

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Proof.

Indeed, from $\alpha = \beta - \gamma$ we derive that each conjugate of α over $\mathbb{Q}$, say $\alpha' \neq \alpha$, is expressible in the form

$$\alpha' = \beta' - \gamma',$$

where β' is a conjugate of the Pisot number β over $\mathbb{Q}$ and γ' is a conjugate of the Pisot number γ over $\mathbb{Q}$.

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where β' is a conjugate of the Pisot number β over $\mathbb{Q}$ and γ' is a conjugate of the Pisot number γ over $\mathbb{Q}$. Clearly, $(\beta', \gamma') \neq (\beta, \gamma)$, since $\alpha' \neq \alpha$. If $\beta' \neq \beta$ and $\gamma' \neq \gamma$, then $|\beta'| < 1$ and $|\gamma'| < 1$, so $\alpha' \in D$.

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Theorem 7

Let α be a real algebraic integer of degree $d \geq 2$ over $\mathbb{Q}$ such that its conjugates, except possibly for α itself, are all in $|z| < 2$. Then, α can be written as a difference of two Pisot numbers. Moreover, both these Pisot numbers can be chosen in the field $\mathbb{Q}(\alpha)$, and both be of degree d over $\mathbb{Q}$.

Its main corollary

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Clearly, if α of degree $d \geq 2$ is a difference of two Pisot numbers $\beta, \gamma \in \mathbb{Q}(\alpha)$ of degree d, then other $d-1$ conjugates of α over $\mathbb{Q}$ must all be of moduli less than 2. Hence, Theorem 7 implies the following:

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Corollary 8

A real algebraic integer α of degree $d \geq 2$ over $\mathbb{Q}$ can be written as a difference of two Pisot numbers of degree d in $\mathbb{Q}(\alpha)$ if and only if the conjugates of α, except possibly for α itself, all lie in $|z| < 2$.

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Corollary 8

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Of course, Corollary 8 immediately implies Theorem 4, where we showed that every Salem number is a difference of two Pisot numbers.

Conjugates outside D

On the other hand, if a real algebraic integer α has some conjugates in a part of the strip S that is outside D, then it is not always expressible as a difference of two Pisot numbers. This may happen already for degree $d = 2$.

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Let α be a real quadratic algebraic integer with conjugate $\alpha' \neq \alpha$ over $\mathbb{Q}$. Then, α is always expressible as a difference of two Pisot numbers except when

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$$\alpha < \alpha' < -2 \tag{2}$$

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or

$$2 < \alpha' < \alpha. \tag{3}$$

Examples

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For example, the number $\alpha = 3 + \sqrt{2}$ is expressible as a difference of two Pisot numbers, since $\alpha' = 3 - \sqrt{2}$, and neither (2) nor (3) holds. (This also follows from Theorem 7 due to $|\alpha'| < 2$.)

For example, the number $\alpha = 3 + \sqrt{2}$ is expressible as a difference of two Pisot numbers, since $\alpha' = 3 - \sqrt{2}$, and neither (2) nor (3) holds. (This also follows from Theorem 7 due to $|\alpha'| < 2$.) However, the number $4 + \sqrt{2}$ cannot be written as a difference of two Pisot numbers, since $\alpha' = 4 - \sqrt{2}$, so that (3) is true.

Algebraic integers of prime degree

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(i) *the conjugates of α, except possibly for α itself, are all in $|z| < 2$;*

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(i) *the conjugates of α, except possibly for α itself, are all in $|z| < 2$;*

(ii) *for some integer $k \geq \max(2, 1-\alpha)$ the conjugates of α, except for α itself, all lie in $|z+k| < 1$;*

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(iii) *for some integer $k \geq \max(2, 1+\alpha)$ the conjugates of α, except for α itself, all lie in $|z-k| < 1$.*

An important tool

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Lemma 11

Let α be a real algebraic number of degree $d \geq 2$ over $\mathbb{Q}$ with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$. Set

$$h(x) = (x - \alpha_2) \dots (x - \alpha_d) = x^{d-1} + b_{d-2}x^{d-2} + \dots + b_1 x + b_0.$$

Then, the numbers $1, b_0, b_1, \dots, b_{d-2}$ are real and linearly independent over $\mathbb{Q}$.

Other ingredients

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Lemma 12

Let $\lambda_1, \ldots, \lambda_N$*, where* $N \in \mathbb{N}$*, be real numbers such that* $1, \lambda_1, \ldots, \lambda_N$ *are* $\mathbb{Q}$*-linearly independent, and let* $\omega_1, \ldots, \omega_N \in \mathbb{R}$*.*

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Let $\lambda_1, \dots, \lambda_N$*, where* $N \in \mathbb{N}$*, be real numbers such that* $1, \lambda_1, \dots, \lambda_N$ *are* $\mathbb{Q}$*-linearly independent, and let* $\omega_1, \dots, \omega_N \in \mathbb{R}$*. Then, for any* $\varepsilon > 0$ *there are infinitely many positive integers* q *such that*

$$\| q\lambda_j - \omega_j \| < \varepsilon$$

for $j = 1, \dots, N$*.*

...and one more lemma

Lemma 13

Assume that $\alpha = \beta - \gamma$ *for some Pisot numbers* β, γ. *If* $|\alpha| > 2$, *then* β *and* γ *are both in the field* $\mathbb{Q}(\alpha)$. *Also, if* $\alpha > 2$, *then no conjugate of* α *over* $\mathbb{Q}$ *can lie in the set*

$$\{z \in \mathbb{C} \,:\, 2 \le |z| < \alpha, \quad \Re(z) > 0\}.$$

Proof of Theorem 3

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Let α be a positive algebraic number of degree d over $\mathbb{Q}$ with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$. The claim is trivial for $d = 1$, since every integer $k \geq 2$ is a Pisot number and every positive rational number is a quotient of two such numbers.

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Fix a positive number $u < 1$ satisfying

$$mu \max(1, |\alpha_2|, \dots, |\alpha_d|) < 1, \tag{4}$$

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Fix a positive number $u < 1$ satisfying

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and a positive number $v > 1$ satisfying

$$mv\alpha > 1. \tag{5}$$

Select a Pisot number $\beta \in \mathbb{Q}(\alpha)$ of degree d. A natural power of β is also a Pisot number of degree d, so by replacing β by its large power if necessary, we can assume that $\beta > v$ and that the other $d - 1$ conjugates of β over $\mathbb{Q}$ are all in $|z| < u$.

Proof of Theorem 3 (continuation)

Write this β in the form $\beta = f(\alpha)$, where f is a nonconstant polynomial of degree at most $d - 1$ with rational coefficients. Then, the numbers $\beta_j = f(\alpha_j)$, $j = 1, \dots, d$, are the conjugates of $\beta = \beta_1$ over $\mathbb{Q}$.

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$$\beta = f(\alpha) > v \quad \text{and} \quad |\beta_j| = |f(\alpha_j)| < u \quad \text{for} \quad j = 2, \dots, d.$$

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$$\beta = f(\alpha) > v \quad \text{and} \quad |\beta_j| = |f(\alpha_j)| < u \quad \text{for} \quad j = 2, \dots, d.$$

We claim that under assumption on the constants $u \in (0,1)$ as in (4) and $v > 1$ as in (5), the numbers $m\alpha\beta \in \mathbb{Q}(\alpha)$ and $m\beta \in \mathbb{Q}(\alpha)$ are both Pisot numbers of degree d. This will complete our proof, since their quotient is α.

Proof of Theorem 3 (continuation)

Firstly, $m\beta$ is a Pisot number, since it is an algebraic integer greater than $m > 1$, whose other conjugates $m\beta_j$, $j = 2, \dots, d$, all lie in $|z| < 1$ by $|\beta_j| < u$ and (4). Of course, $m\beta \in \mathbb{Q}(\alpha)$ is of degree d over $\mathbb{Q}$, since so is β.

Proof of Theorem 3 (continuation)

Firstly, $m\beta$ is a Pisot number, since it is an algebraic integer greater than $m > 1$, whose other conjugates $m\beta_j$, $j = 2, \ldots, d$, all lie in $|z| < 1$ by $|\beta_j| < u$ and (4). Of course, $m\beta \in \mathbb{Q}(\alpha)$ is of degree d over $\mathbb{Q}$, since so is β.

Secondly, the number $m\alpha\beta = m\alpha f(\alpha) \in \mathbb{Q}(\alpha)$ is a positive algebraic integer, since so are $m\alpha$ and β. It is greater than 1 by $\beta > v$ and (5). Its other conjugates are $m\alpha_j f(\alpha_j) = m\alpha_j\beta_j$, $j = 2, \ldots, d$. They are all in $|z| < 1$ due to $|\beta_j| < u$ and (4).

Proof of Theorem 3 (continuation)

Hence, $m\alpha f(\alpha) \in \mathbb{Q}(\alpha)$ is a Pisot number of degree d over $\mathbb{Q}$ by

Lemma 14

Let α be a real algebraic number of degree $d \geq 2$ with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ over $\mathbb{Q}$, and let f be a nonconstant polynomial with rational coefficients such that $f(\alpha) > 0$ and $|f(\alpha_j)| < 1$ for $j = 2, \dots, d$. If $f(\alpha) \in \mathbb{Q}(\alpha)$ is an algebraic integer, then it is a Pisot number of degree d.

applied to the polynomial $mxf(x) \in \mathbb{Q}[x]$.

Proof of Theorem 3 (continuation)

Hence, $m\alpha f(\alpha) \in \mathbb{Q}(\alpha)$ is a Pisot number of degree d over $\mathbb{Q}$ by

Lemma 14

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Therefore, $m\alpha\beta \in \mathbb{Q}(\alpha)$ and $m\beta \in \mathbb{Q}(\alpha)$ indeed are both Pisot numbers of degree d, which finishes the proof.