Numbers expressible by quotients or differences of two Pisot numbers

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Definitions

An algebraic integer $\alpha > 1$ is a Pisot number if its conjugates over $\mathbb{Q}$ (if any) all lie in the open unit disc $|z| < 1$.

An algebraic integer $\alpha > 1$ of degree $d \geq 4$ is a Salem number if one of its conjugates over $\mathbb{Q}$ is $\alpha - 1$ and the remaining $d - 2$ conjugates are all on the circle $|z| = 1$.

Pisot and Salem numbers appear in various contexts of number theory, harmonic analysis, dynamical systems, tilings, etc.
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Pisot numbers

The set of Pisot numbers is quite well understood. In R. Salem, Power series with integer coefficients, Duke Math. J., 12 (1945), 153–172, Salem showed that it is an infinite closed set, while Siegel C. L. Siegel, Algebraic integers whose conjugates lie in the unit circle, Duke Math. J., 11 (1944), 597–602, proved that its smallest element is the real root \( \theta = 1.32471... \) of \( x^3 - x - 1 \).
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Earlier results on Pisot numbers

In 1945, Salem himself proved that

Theorem 1

Every Salem number is expressible as a quotient of two Pisot numbers.

On the other hand, in A. Dubickas, Mahler measures generate the largest possible groups, Math. Res. Lett. 11 (2004), 279–283. I showed that

Theorem 2

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Mahler measure

Recall that the Mahler measure $M(\alpha)$ of a nonzero algebraic number $\alpha$ is the modulus of the product of its conjugates lying outside the unit circle and the leading coefficient of its minimal polynomial in $\mathbb{Z}[x]$.

Thus, for a real algebraic number $\alpha > 1$ we have $M(\alpha) \geq \alpha$ with equality if and only if $\alpha$ is a Salem or a Pisot number.
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Every positive algebraic number is a quotient of Pisot numbers

Therefore, the following theorem, recently proved in A. Dubickas, Every Salem number is a difference of two Pisot numbers, Proc. Edinb. Math. Soc., 66 (2023), 862–867, generalizes both Theorem 1 and Theorem 2:

Theorem 3
Every real positive algebraic number \( \alpha \) of degree \( d \) is expressible as a quotient of two Pisot numbers of degree \( d \) from the field \( \mathbb{Q}(\alpha) \).
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In all what follows we will study the additive version of this problem.

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**Theorem 4**
Every Salem number is expressible as a difference of two Pisot numbers.

More explicitly, I showed that

**Theorem 5**
For each Salem number $\alpha$ of degree $d \geq 4$ there exist infinitely many $n \in \mathbb{N}$ for which $\alpha^2n - 1 - \alpha n + \alpha$ and $\alpha^2n - 1 - \alpha n - \alpha$ are both Pisot numbers of degree $d$. The smallest such $n$ is at most $\frac{6d}{2 - 1}$. 

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A more general setting of the problem

We are interested which algebraic numbers $\alpha$ are expressible in the form of the difference $\alpha = \beta - \gamma$ (1) of some Pisot numbers $\beta$ and $\gamma$.

It is clear that each such $\alpha$ must be a real algebraic integer since so are $\beta$ and $\gamma$.

As we will show below, not every real algebraic integer $\alpha$ is expressible as a difference of two Pisot numbers due to several restrictions on the conjugates of $\alpha$. 
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As we will show below, not every real algebraic integer $\alpha$ is expressible as a difference of two Pisot numbers due to several restrictions on the conjugates of $\alpha$. 
Both $\beta$ and $\gamma$ should be in the same field as $\alpha$.

Theorem 6

If $\alpha$ is expressible as a difference of two Pisot numbers, then there exist Pisot numbers $\beta, \gamma \in \mathbb{Q}(\alpha)$ for which $\alpha = \beta - \gamma$.

This, and other results below, is in:

Both $\beta$ and $\gamma$ should be in the same field as $\alpha$.

Our next theorem shows that, under the assumption that the representation (1) is possible, the Pisot numbers $\beta$ and $\gamma$ in it can always be chosen to be in the field $\mathbb{Q}(\alpha)$.

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This, and other results below, is in:

Let \( d \) be the degree of \( \alpha \) over \( \mathbb{Q} \). The situation is very simple for \( d = 1 \). Then, \( \alpha \in \mathbb{Z} \) and we clearly have \( \alpha = (m + \alpha) - m \), where \( m \) is a positive integer satisfying \( m \geq \max(2, 2 - \alpha) \). Observe that \( m + \alpha \) and \( m \) are both rational integers greater than or equal to 2, so both are Pisot numbers. Of course, other representations are also possible. For example, \( \alpha = 1 \) is the difference of two quadratic Pisot numbers \( 2 + \sqrt{2} \) and \( 1 + \sqrt{2} \) that are outside the field \( \mathbb{Q}(\alpha) = \mathbb{Q} \).
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In all that follows we, therefore, assume that $d \geq 2$. We now show that the conjugates of $\alpha$ expressible as a difference of two Pisot numbers as in (1) must all lie in the union of the open disc $D = \{z \in \mathbb{C} : \text{divides.alt0} z \text{divides.alt0} < 2\}$ and the infinite strip $S = \{z \in \mathbb{C} : \text{divides.alt0} I z \text{divides.alt0} < 1\}$. 

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Quotients & differences of two Pisot numbers
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$$D = \{z \in \mathbb{C} : |z| < 2\}$$

and the infinite strip

$$S = \{z \in \mathbb{C} : |\Im z| < 1\}.$$
Degree $d$ (continuation)

Proof. Indeed, from $\alpha = \beta - \gamma$ we derive that each conjugate of $\alpha$ over $\mathbb{Q}$, say $\alpha' \neq \alpha$, is expressible in the form $\alpha' = \beta' - \gamma'$, where $\beta'$ is a conjugate of the Pisot number $\beta$ over $\mathbb{Q}$ and $\gamma'$ is a conjugate of the Pisot number $\gamma$ over $\mathbb{Q}$.

Clearly, $(\beta', \gamma') \neq (\beta, \gamma)$, since $\alpha' \neq \alpha$. If $\beta' \neq \beta$ and $\gamma' \neq \gamma$, then $\beta' < 1$ and $\gamma' < 1$, so $\alpha' \in D$.

In the case when either $\beta' = \beta$ (but $\gamma' \neq \gamma$) or $\gamma' = \gamma$ (but $\beta' \neq \beta$) the imaginary part of $\alpha'$ is either $-I(\gamma')$ or $I(\beta')$.

Therefore, since $\beta', \gamma' \in \{z \in \mathbb{C} : z < 1\}$, both numbers $-I(\gamma')$, $I(\beta')$ are in $(-1, 1)$, which implies $\alpha' \in S$.

Consequently, $\alpha' \in D \cup S$, as claimed. Also, $\alpha \in \mathbb{R} \subset S$.

Thus, under assumption (1), the conjugates of $\alpha$ over $\mathbb{Q}$ (including $\alpha$ itself) must all lie in $D \cup S$. 
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Our main theorem shows that every real algebraic integer $\alpha$, whose conjugates, except possibly for $\alpha$ itself, all lie in $D$, is expressible as a difference of two Pisot numbers.

Theorem 7

Let $\alpha$ be a real algebraic integer of degree $d \geq 2$ over $\mathbb{Q}$ such that its conjugates, except possibly for $\alpha$ itself, are all in $\mathbb{Q}(\alpha)$. Then, $\alpha$ can be written as a difference of two Pisot numbers. Moreover, both these Pisot numbers can be chosen in the field $\mathbb{Q}(\alpha)$, and both be of degree $d$ over $\mathbb{Q}$. 

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Its main corollary

Clearly, if $\alpha$ of degree $d \geq 2$ is a difference of two Pisot numbers $\beta, \gamma \in \mathbb{Q}(\alpha)$ of degree $d$, then other $d - 1$ conjugates of $\alpha$ over $\mathbb{Q}$ must all be of moduli less than 2. Hence, Theorem 7 implies the following:

**Corollary 8**

A real algebraic integer $\alpha$ of degree $d \geq 2$ over $\mathbb{Q}$ can be written as a difference of two Pisot numbers of degree $d$ in $\mathbb{Q}(\alpha)$ if and only if the conjugates of $\alpha$, except possibly for $\alpha$ itself, all lie in $\mathbb{Q}$.

Of course, Corollary 8 immediately implies Theorem 4, where we showed that every Salem number is a difference of two Pisot numbers.
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Conjugates outside $D$

On the other hand, if a real algebraic integer $\alpha$ has some conjugates in a part of the strip $S$ that is outside $D$, then it is not always expressible as a difference of two Pisot numbers. This may happen already for degree $d = 2$.

The next theorem gives a full characterization of real quadratic algebraic integers $\alpha$ expressible as a difference of two Pisot numbers.

**Theorem 9**

Let $\alpha$ be a real quadratic algebraic integer with conjugate $\alpha' \neq \alpha$ over $\mathbb{Q}$. Then, $\alpha$ is always expressible as a difference of two Pisot numbers except when $\alpha < \alpha' < -2$ (2) or $2 < \alpha' < \alpha$.

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On the other hand, if a real algebraic integer \( \alpha \) has some conjugates in a part of the strip \( S \) that is outside \( D \), then it is not always expressible as a difference of two Pisot numbers. This may happen already for degree \( d = 2 \).

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\[
\alpha < \alpha' < -2 \quad (2)
\]

or

\[
2 < \alpha' < \alpha. \quad (3)
\]
Examples

For example, the number $\alpha = 3 + \sqrt{2}$ is expressible as a difference of two Pisot numbers, since $\alpha' = 3 - \sqrt{2}$, and neither (2) nor (3) holds. (This also follows from Theorem 7 due to $\alpha' < 2$.)

However, the number $4 + \sqrt{2}$ cannot be written as a difference of two Pisot numbers, since $\alpha' = 4 - \sqrt{2}$, so that (3) is true.
For example, the number $\alpha = 3 + \sqrt{2}$ is expressible as a difference of two Pisot numbers, since $\alpha' = 3 - \sqrt{2}$, and neither (2) nor (3) holds. (This also follows from Theorem 7 due to $|\alpha'| < 2$.)
For example, the number $\alpha = 3 + \sqrt{2}$ is expressible as a difference of two Pisot numbers, since $\alpha' = 3 - \sqrt{2}$, and neither (2) nor (3) holds. (This also follows from Theorem 7 due to $|\alpha'| < 2$.) However, the number $4 + \sqrt{2}$ cannot be written as a difference of two Pisot numbers, since $\alpha' = 4 - \sqrt{2}$, so that (3) is true.
Algebraic integers of prime degree

Theorem 10

Let $\alpha$ be a real algebraic integer of degree $d$ over $\mathbb{Q}$. If $d$ is a prime number, then $\alpha$ is a difference of two Pisot numbers if and only if one of the following is true:

(i) the conjugates of $\alpha$, except possibly for $\alpha$ itself, are all in $\mathbb{Q} / \mathbb{Z} < 2$;

(ii) for some integer $k \geq \max(2, 1 - \alpha)$ the conjugates of $\alpha$, except for $\alpha$ itself, all lie in $\mathbb{Q} / \mathbb{Z} + k < 1$;

(iii) for some integer $k \geq \max(2, 1 + \alpha)$ the conjugates of $\alpha$, except for $\alpha$ itself, all lie in $\mathbb{Q} / \mathbb{Z} - k < 1$. 

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Quotients & differences of two Pisot numbers
The next theorem characterizes all algebraic integers of prime degree expressible as a difference of two Pisot numbers.
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2. For some integer $k \geq \max(2, 1 - \alpha)$, the conjugates of $\alpha$, except for $\alpha$ itself, all lie in $|z + k| < 1$;
3. For some integer $k \geq \max(2, 1 + \alpha)$, the conjugates of $\alpha$, except for $\alpha$ itself, all lie in $|z - k| < 1$.

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An important tool

An important ingredient in the proof of Theorem 7 is the following assertion (which holds for any algebraic number):

**Lemma 11**

Let \( \alpha \) be a real algebraic number of degree \( d \geq 2 \) over \( \mathbb{Q} \) with conjugates \( \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d \). Set

\[
h(x) = (x - \alpha_2) \cdots (x - \alpha_d) = x^d - b_{d-2}x^{d-2} + \cdots + b_1x + b_0.
\]

Then, the numbers \( 1, b_0, b_1, \ldots, b_{d-2} \) are real and linearly independent over \( \mathbb{Q} \).
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Then, the numbers $1, b_0, b_1, \ldots, b_{d-2}$ are real and linearly independent over $\mathbb{Q}$.
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Then, the numbers $1, b_0, b_1, \ldots, b_{d-2}$ are real and linearly independent over $\mathbb{Q}$. 

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Quotients & differences of two Pisot numbers
Kronecker's approximation theorem:

Lemma 12

Let $\lambda_1, \ldots, \lambda_N$, where $N \in \mathbb{N}$, be real numbers such that $1, \lambda_1, \ldots, \lambda_N$ are $\mathbb{Q}$-linearly independent, and let $\omega_1, \ldots, \omega_N \in \mathbb{R}$.

Then, for any $\varepsilon > 0$ there are infinitely many positive integers $q$ such that

$$\|q \lambda_j - \omega_j\| < \varepsilon$$

for $j = 1, \ldots, N$. 

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Quotients & differences of two Pisot numbers
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Let \( \lambda_1, \ldots, \lambda_N \), where \( N \in \mathbb{N} \), be real numbers such that \( 1, \lambda_1, \ldots, \lambda_N \) are \( \mathbb{Q} \)-linearly independent, and let \( \omega_1, \ldots, \omega_N \in \mathbb{R} \). Then, for any \( \varepsilon > 0 \) there are infinitely many positive integers \( q \) such that \( q \lambda_j - \omega_j \) is closer to 0 than \( \varepsilon \) for \( j = 1, \ldots, N \).
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\[
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...and one more lemma

Lemma 13

Assume that $\alpha = \beta - \gamma$ for some Pisot numbers $\beta, \gamma$. If $\alpha > 2$, then $\beta$ and $\gamma$ are both in the field $\mathbb{Q}(\alpha)$. Also, if $\alpha > 2$, then no conjugate of $\alpha$ over $\mathbb{Q}$ can lie in the set \( \{ z \in \mathbb{C} : 2 \leq \frac{\text{divides}(z)}{\text{divides}(\alpha)} < \alpha, R(z) > 0 \} \).
Lemma 13

Assume that $\alpha = \beta - \gamma$ for some Pisot numbers $\beta, \gamma$. If $|\alpha| > 2$, then $\beta$ and $\gamma$ are both in the field $\mathbb{Q}(\alpha)$. Also, if $\alpha > 2$, then no conjugate of $\alpha$ over $\mathbb{Q}$ can lie in the set

$$\{ z \in \mathbb{C} : 2 \leq |z| < \alpha, \quad \Re(z) > 0 \}.$$
Proof of Theorem 3

Let $\alpha$ be a positive algebraic number of degree $d$ over $\mathbb{Q}$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. The claim is trivial for $d = 1$, since every integer $k \geq 2$ is a Pisot number and every positive rational number is a quotient of two such numbers.

Assume that $d \geq 2$, and let $m$ be a positive integer for which $m \alpha$ is an algebraic integer.

Fix a positive number $u < 1$ satisfying $mu \max(1, 1/\divides.alt0 \alpha_2/\divides.alt0, \ldots, 1/\divides.alt0 \alpha_d/\divides.alt0) < 1$, (4)

and a positive number $v > 1$ satisfying $mv \alpha > 1$. (5)

Select a Pisot number $\beta \in \mathbb{Q}(\alpha)$ of degree $d$. A natural power of $\beta$ is also a Pisot number of degree $d$, so by replacing $\beta$ by its large power if necessary, we can assume that $\beta > v$ and that the other $d - 1$ conjugates of $\beta$ over $\mathbb{Q}$ are all in $\divides.alt0 z/\divides.alt0 < u$. (76x3)
Proof of Theorem 3

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Select a Pisot number $\beta \in \mathbb{Q}(\alpha)$ of degree $d$. A natural power of $\beta$ is also a Pisot number of degree $d$, so by replacing $\beta$ by its large power if necessary, we can assume that $\beta > v$ and that the other $d - 1$ conjugates of $\beta$ over $\mathbb{Q}$ are all in $|z| < u$. 

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Quotients & differences of two Pisot numbers
Proof of Theorem 3 (continuation)

Write this \( \beta \) in the form \( \beta = f(\alpha) \), where \( f \) is a nonconstant polynomial of degree at most \( d - 1 \) with rational coefficients.

Then, the numbers \( \beta_j = f(\alpha_j) \), \( j = 1, \ldots, d \), are the conjugates of \( \beta = \beta_1 \) over \( \mathbb{Q} \).

Recall that, by the choice of \( \beta \), we have \( \beta = f(\alpha) > v \) and \( \beta_j \mid f(\alpha_j) < u \) for \( j = 2, \ldots, d \).

We claim that under assumption on the constants \( u \in (0,1) \) as in (4) and \( v > 1 \) as in (5), the numbers \( m_{\alpha \beta} \in \mathbb{Q}(\alpha) \) and \( m_{\beta} \in \mathbb{Q}(\alpha) \) are both Pisot numbers of degree \( d \). This will complete our proof, since their quotient is \( \alpha \).
Write this $\beta$ in the form $\beta = f(\alpha)$, where $f$ is a nonconstant polynomial of degree at most $d - 1$ with rational coefficients. Then, the numbers $\beta_j = f(\alpha_j)$, $j = 1, \ldots, d$, are the conjugates of $\beta = \beta_1$ over $\mathbb{Q}$. 

Recall that, by the choice of $\beta$, we have $\beta > v$ and $\beta_j \divides f(\alpha_j) < u$ for $j = 2, \ldots, d$.

We claim that under assumption on the constants $u \in (0, 1)$ as in (4) and $v > 1$ as in (5), the numbers $m_{\alpha} \beta \in \mathbb{Q}(\alpha)$ and $m_{\beta} \alpha \in \mathbb{Q}(\alpha)$ are both Pisot numbers of degree $d$. This will complete our proof, since their quotient is $\alpha$. 

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Quotients & differences of two Pisot numbers
Write this $\beta$ in the form $\beta = f(\alpha)$, where $f$ is a nonconstant polynomial of degree at most $d - 1$ with rational coefficients. Then, the numbers $\beta_j = f(\alpha_j)$, $j = 1, \ldots, d$, are the conjugates of $\beta = \beta_1$ over $\mathbb{Q}$. Recall that, by the choice of $\beta$, we have

$$\beta = f(\alpha) > v \quad \text{and} \quad |\beta_j| = |f(\alpha_j)| < u \quad \text{for} \quad j = 2, \ldots, d.$$
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We claim that under assumption on the constants $u \in (0, 1)$ as in (4) and $v > 1$ as in (5), the numbers $m\alpha\beta \in \mathbb{Q}(\alpha)$ and $m\beta \in \mathbb{Q}(\alpha)$ are both Pisot numbers of degree $d$. This will complete our proof, since their quotient is $\alpha$. 

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Quotients & differences of two Pisot numbers
Proof of Theorem 3 (continuation)

Firstly, $m^\beta$ is a Pisot number, since it is an algebraic integer greater than $m^\beta > 1$, whose other conjugates $m^\beta_j$, $j = 2, \ldots, d$, all divide $z < 1$ by (4). Of course, $m^\beta \in \mathbb{Q}(\alpha)$ is of degree $d$ over $\mathbb{Q}$, since so is $\beta$.

Secondly, the number $m^\alpha \beta = m^\alpha f(\alpha) \in \mathbb{Q}(\alpha)$ is a positive algebraic integer, since so are $m^\alpha$ and $\beta$. It is greater than 1 by $\beta > v$ and (5). Its other conjugates are $m^\alpha_j f(\alpha_j) = m^\alpha_j \beta_j$, $j = 2, \ldots, d$. They are all in $z < 1$ due to $\beta_j < u$ and (4).
Firstly, $m\beta$ is a Pisot number, since it is an algebraic integer greater than $m > 1$, whose other conjugates $m\beta_j$, $j = 2, \ldots, d$, all lie in $|z| < 1$ by $|\beta_j| < u$ and (4). Of course, $m\beta \in \mathbb{Q}(\alpha)$ is of degree $d$ over $\mathbb{Q}$, since so is $\beta$. 
Firstly, $m \beta$ is a Pisot number, since it is an algebraic integer greater than $m > 1$, whose other conjugates $m \beta_j$, $j = 2, \ldots, d$, all lie in $|z| < 1$ by $|\beta_j| < u$ and (4). Of course, $m \beta \in \mathbb{Q}(\alpha)$ is of degree $d$ over $\mathbb{Q}$, since so is $\beta$.

Secondly, the number $m \alpha \beta = m \alpha f(\alpha) \in \mathbb{Q}(\alpha)$ is a positive algebraic integer, since so are $m \alpha$ and $\beta$. It is greater than 1 by $\beta > v$ and (5). Its other conjugates are $m \alpha_j f(\alpha_j) = m \alpha_j \beta_j$, $j = 2, \ldots, d$. They are all in $|z| < 1$ due to $|\beta_j| < u$ and (4).
Proof of Theorem 3 (continuation)

Hence,

\[ m_\alpha f(\alpha) \in Q(\alpha) \]

is a Pisot number of degree \( d \) over \( \mathbb{Q} \) by Lemma 14.

Let \( \alpha \) be a real algebraic number of degree \( d \geq 2 \) with conjugates \( \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d \) over \( \mathbb{Q} \), and let \( f \) be a nonconstant polynomial with rational coefficients such that \( f(\alpha_j) > 0 \) and \( \text{divides}.alt0 - f(\alpha_j) < 1 \) for \( j = 2, \ldots, d \). If \( f(\alpha) \in Q(\alpha) \) is an algebraic integer, then it is a Pisot number of degree \( d \), which finishes the proof.
Hence, $m\alpha f(\alpha) \in \mathbb{Q}(\alpha)$ is a Pisot number of degree $d$ over $\mathbb{Q}$ by Lemma 14.

**Lemma 14**

Let $\alpha$ be a real algebraic number of degree $d \geq 2$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ over $\mathbb{Q}$, and let $f$ be a nonconstant polynomial with rational coefficients such that $f(\alpha) > 0$ and $|f(\alpha_j)| < 1$ for $j = 2, \ldots, d$. If $f(\alpha) \in \mathbb{Q}(\alpha)$ is an algebraic integer, then it is a Pisot number of degree $d$.

applied to the polynomial $mxf(x) \in \mathbb{Q}[x]$. 
Hence, \(m\alpha f(\alpha) \in \mathbb{Q}(\alpha)\) is a Pisot number of degree \(d\) over \(\mathbb{Q}\) by Lemma 14.

**Lemma 14**

Let \(\alpha\) be a real algebraic number of degree \(d \geq 2\) with conjugates \(\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d\) over \(\mathbb{Q}\), and let \(f\) be a nonconstant polynomial with rational coefficients such that \(f(\alpha) > 0\) and \(|f(\alpha_j)| < 1\) for \(j = 2, \ldots, d\). If \(f(\alpha) \in \mathbb{Q}(\alpha)\) is an algebraic integer, then it is a Pisot number of degree \(d\).

applied to the polynomial \(mxf(x) \in \mathbb{Q}[x]\).

Therefore, \(m\alpha \beta \in \mathbb{Q}(\alpha)\) and \(m\beta \in \mathbb{Q}(\alpha)\) indeed are both Pisot numbers of degree \(d\), which finishes the proof.