Golden numeration systems

Michel Dekking

Numeration 2024

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Short Abstract:
How do the expansions in base phi look like?
What is the relation with the Zeckendorf expansions?
I will give answers to these questions in my talk.

Golden numeration systems

D.: The structure of Zeckendorf expansions, INTEGERS 21, #A6 (2021), 1–10.

D.: The structure of base phi expansions, INTEGERS 24, #A24 (2024), 1–28.

Golden numeration systems

D.: The structure of Zeckendorf expansions, INTEGERS 21, #A6 (2021), 1–10.

D.: The structure of base phi expansions, INTEGERS 24, #A24 (2024), 1–28.

There are hundreds of papers on these systems.

Special recent progress

Jeffrey Shallit and coworkers using Walnut:

Jeffrey O. Shallit, Sonja Linghui Shan: A General Approach to Proving Properties of Fibonacci Representations via Automata Theory. AFL 2023: 228-242 (2023)

Jeffrey O. Shallit: Proving Results About OEIS Sequences with Walnut. CICM 2023: 270-282 (2023)

Jeffrey O. Shallit: Note on a Fibonacci parity sequence. Cryptogr. Commun. 15(2): 309-315 (2023)

Jeffrey O. Shallit: Proving Properties of φ-Representations with the Walnut Theorem-Prover (2024)

and many more.....

Base phi representation

A natural number N is written in base phi if

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed.

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Similarly to base 10 numbers, we write these representations as

$$\beta(N) = d_L d_{L-1} \ldots d_1 d_0 \cdot d_{-1} d_{-2} \ldots d_{R+1} d_R.$$

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$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

The convention is that we are ignoring leading and trailing zeroes.

THEOREM The base phi representation of N is unique.

Base phi representation Example

$$\beta(2) = 10{\cdot}01$$

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Check:

$$\varphi + \frac{1}{\varphi^2} =$$
$$\varphi + \frac{1}{1+\varphi} =$$
$$\frac{\varphi + \varphi^2 + 1}{1+\varphi} =$$
$$\frac{\varphi + \varphi + 1 + 1}{1+\varphi} = 2.$$

Zeckendorf representations

Let $F_0 = 0,\ F_1 = 1,\ F_2 = 1, \ldots$ be the Fibonacci numbers.

Ignoring leading zeros, any natural number N can be written uniquely as

$$N = \sum_{i=0}^{\infty} d_i F_{i+2},$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed.

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We write $Z(N) = d_L \ldots d_0$.

EXAMPLE $Z(6) = 1001$, since $F_5 = 5, F_2 = 1$.

Zeckendorf and base phi

N	$Z(N)$	$\beta(N)$
1	1	1·
2	10	10·01
3	100	100·01
4	101	101·01
5	1000	1000·1001
6	1001	1010·0001
7	1010	10000·0001
8	10000	10001·0001
9	10001	10010·0101
10	10010	10100·0101
11	10100	10101·0101
12	10101	100000·101001
13	100000	100010·001001
14	100001	100100·001001
15	100010	100101·001001

Splitting the base phi expansion

We define

$$\beta^+(N) = d_L d_{L-1} \cdots d_1 d_0.$$

$$\beta^-(N) = d_{-1} d_{-2} \cdots d_{R+1} d_R.$$

So $\beta(N) = \beta^+(N) \cdot \beta^-(N)$.

Connecting Zeckendorf and base phi [1]

N	$Z(N)$	$\beta^+(N)$	$\beta^-(N)$
1	1	1	·
2	10	10	·01
3	100	100	·01
4	101	101	·01
5	1000	1000	·1001
6	1001	1010	·0001
7	1010	10000	·0001
8	10000	10001	·0001
9	10001	10010	·0101
10	10010	10100	·0101
11	10100	10101	·0101
12	10101	100000	·101001
13	100000	100010	·001001
14	100001	100100	·001001
15	100010	100101	·001001

Connecting Zeckendorf and base phi [2]

N	$Z(N)$	$\beta^+(N)$	$\beta^-(N)$
1	1	1	·
2	10	10	·01
3	100	100	·01
4	101	101	·01
5	1000	1000	·1001
6	1001	1010	·0001
7	1010	10000	·0001
8	10000	10001	·0001
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Connecting Zeckendorf and base phi [3]

N	$Z(N)$	$\beta^+(N)$	$\beta^-(N)$
1	1	1	·
2	10	10	·01
3	100	100	·01
4	101	101	·01
5	1000	1000	·1001
6	1001	1010	·0001
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Connecting Zeckendorf and base phi [4]

N	$Z(N)$	$\beta^+(N)$	$\beta^-(N)$
1	1	1	·
2	10	10	·01
3	100	100	·01
4	101	101	·01
5	1000	1000	·1001
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Connecting Zeckendorf and base phi [5]

N	$Z(N)$	$\beta^{+}(N)$	$\beta^{-}(N)$
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4	101	101	·01
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Connecting Zeckendorf and base phi [6]

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Connecting Zeckendorf and base phi: how?

What are the missing words $1001, 100001, \ldots$?

Connecting Zeckendorf and base phi: how?

What are the missing words $1001, 100001, \ldots$?

ANSWER: These are all the words with suffix $10^{2m}1$, for some $m = 1, 2, \ldots$.

For example 101001 is skipped. This is $Z(19)$.

Next 1001001, which is $Z(27)$.

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ANSWER:
$(V_{\text{skip}}(N)) = (6, 14, 19, 27, 35, 40, 48, 53, 61, 69, 74, \dots)$.
I will show: $V_{\text{skip}}(N) = 3\lfloor \varphi N \rfloor + 2N + 1$, where $N = 1, 2, \dots$

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Allouche & Dekking call this a generalized Beatty sequence.

In general: $V(p, q, r) = p\lfloor \varphi N \rfloor + qN + r$.

Connecting Zeckendorf and base phi: how [2]

Consider the sequence of first order differences of $(V_{\rm skip}(N)) = (6, 14, 19, 27, 35, 40, 48, 53, 61, 69, 74, \dots)$:

$$\Delta V_{\rm skip} = 8, 5, 8, 8, 5, 8, 5, 8, 8, 5, \dots$$

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Do we recognize this?

Connecting Zeckendorf and base phi: how [2]

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Do we recognize this?

The Fibonacci word on the alphabet $\{8, 5)$!

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This is a general property of generalized Beatty sequences:

LEMMA GBS Let $V = (V_n)_{n\geq 1}$ be the generalized Beatty sequence defined by $V_n = p\lfloor n\varphi\rfloor + qn + r$, and let ΔV be the sequence of its first differences. Then ΔV is the Fibonacci word over the alphabet $\{2p+q, p+q\}$. Conversely, if $x_{a,b}$ is the Fibonacci word over the alphabet $\{a, b\}$, then every V with $\Delta V = x_{a,b}$ is a generalized Beatty sequence $V = V(a-b, 2b-a, r)$ for some integer r.

Connecting Zeckendorf and base phi: how [2]

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Do we recognize this?

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In the ΔV_{skip} case: $p = 3, q = 2$.

Connecting Zeckendorf and base phi: how [3]

But, how do you prove that $V_{\mathrm{skip}}(N) = 3\lfloor \varphi N \rfloor + 2N + 1$?

Connecting Zeckendorf and base phi: how [3]

But, how do you prove that $V_{\mathrm{skip}}(N) = 3\lfloor \varphi N \rfloor + 2N + 1$?

For this one analyses the skipping process.

The essential ingredient in this analysis is the following result from D: Base phi representations and golden mean beta-expansions, Fibonacci Quart. 58 (2020).

PROPOSITION Let $\beta(N) = (d_i(N))$ be the base phi expansion of N. Then

$d_1 d_0 \cdot d_{-1}(N) = 10 \cdot 1$ never occurs,

$d_1 d_0 \cdot d_{-1}(N) = 00 \cdot 1 \Leftrightarrow N = 3\lfloor n\varphi \rfloor + n + 1$ for some natural number n.

Connecting Zeckendorf and base phi: how [4]

Step by step: $Z(N) \mapsto Z(N+1)$ and $\beta(N') \mapsto \beta(N'+1)$....

Zeckendorf

Base golden mean

The case $d_{-1} = 0$, $d_0 = 0$

Connecting Zeckendorf and base phi: how [4]

Step by step: $Z(N) \mapsto Z(N+1)$ and $\beta(N') \mapsto \beta(N'+1)$....

Zeckendorf	Base golden mean
	The case $d_{-1} = 0$, $d_0 = 0$
$Z(N) = d_L \cdots d_2 d_1\, 0$	$\beta(N') = d_L \cdots d_1\, 0 \cdot 0\, d_{-2} \cdots$
1	$1 \cdot 0$
$+$	$+$
$Z(N+1) = d_L \cdots d_2 d_1\, 1$	$\beta(N'+1) = d_L \cdots d_1\, 1 \cdot 0\, d_{-2} \cdots$

N.B. If $d_1 = 1$, then both $Z(N+1)$ and $\beta^+(N'+1)$ would end in 11. One has to get rid of this by applying a number of times $011 \mapsto 100$, both for Z and for β. This results in equal words $Z(N+1)$ and $\beta^+(N'+1)$.

Connecting Zeckendorf and base phi: how [5]

Zeckendorf

Base golden mean

The case $d_{-1} = 0$, $d_0 = 1$

Connecting Zeckendorf and base phi: how [5]

Zeckendorf	Base golden mean
	The case $d_{-1} = 0$, $d_0 = 1$

$$Z(N) = d_L \cdots d_2\, 0\, 1 \qquad \beta(N') = d_L \cdots d_2\, 0\, 1 \cdot 0\, d_{-2} \cdots$$

$$1 \qquad 1 \cdot 0$$

$$\underline{\qquad\qquad\qquad} \; + \qquad \underline{\qquad\qquad\qquad} \; +$$

$$Z(N+1) = d_L \cdots d_2\, 1\, 0 \qquad \beta(N'+1) = d_L \cdots d_2\, 0\, 2 \cdot 0\, d_{-2} \cdots$$

$$\beta(N'+1) = d_L \cdots d_2\, 1\, 0 \cdot 0\, (d_{-2}+1) \cdots$$

Connecting Zeckendorf and base phi: how [6]

Zeckendorf

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The case $d_{-1} = 1 \Rightarrow d_0 = 0, d_1 = 0$

Connecting Zeckendorf and base phi: how [6]

Zeckendorf	Base golden mean
	The case $d_{-1} = 1 \Rightarrow d_0 = 0, d_1 = 0$

$$
\begin{array}{rr}
Z(N) = d_L \cdots d_2\, 0\, 0 & \\
1 & \\
\hline
Z(N+1) = d_L \cdots d_2\, 0\, 1 & +
\end{array}
\qquad
\begin{array}{rr}
\beta(N') = d_L \cdots d_2\, 0\, 0 \cdot 1\, d_{-2} \cdots & \\
1 \cdot 0 & \\
\hline
\beta(N'+1) = d_L \cdots d_2\, 0\, 1 \cdot 1\, d_{-2} \cdots & + \\
\beta(N'+1) = d_L \cdots d_2\, 1\, 0 \cdot 0\, d_{-2} \cdots &
\end{array}
$$

Connecting Zeckendorf and base phi: how [6]

Zeckendorf

Base golden mean

The case $d_{-1} = 1 \Rightarrow d_0 = 0, d_1 = 0$

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$$1 \qquad 1 \cdot 0$$

$$+ \qquad +$$

$$Z(N+1) = d_L \cdots d_2\, 0\, 1 \qquad \beta(N'+1) = d_L \cdots d_2\, 0\, 1 \cdot 1\, d_{-2} \cdots$$

$$\beta(N'+1) = d_L \cdots d_2\, 1\, 0 \cdot 0\, d_{-2} \cdots$$

$Z(N+1) = d_L \cdots d_2\, 0\, 1$ has been skipped!

Where are the $d_{-1} = 1$?

We have seen: skipping happens exactly at the $d_{-1} = 1$.

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Now recall:

PROPOSITION Let $\beta(N) = (d_i(N))$ be the base phi expansion of N. Then

$d_1 d_0 \cdot d_{-1}(N) = 10 \cdot 1$ never occurs,

$d_1 d_0 \cdot d_{-1}(N) = 00 \cdot 1 \Leftrightarrow N = 3\lfloor \varphi n \rfloor + n + 1$ for some natural number n.

With **LEMMA GBS** this directly yields that

$V_{\text{skip}}(N) = 3\lfloor \varphi N \rfloor + 2N + 1.$

Base phi and the Lucas numbers

The Lucas numbers $(L_n) = (2, 1, 3, 4, 7, 11, 18, 29, \dots)$:

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2.$$

From $L_{2n} = \varphi^{2n} + \varphi^{-2n}$, and $L_{2n+1} = L_{2n} + L_{2n-1}$:

$$\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1} 1, \quad \beta(L_{2n+1}) = 1(01)^n \cdot (01)^n.$$

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$$\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1} 1, \quad \beta(L_{2n+1}) = 1(01)^n \cdot (01)^n.$$

The basis for proving properties of base phi

RECURSIVE STRUCTURE THEOREM

Let the odd and even Lucas intervals be given by

$$\Lambda_{2n+1} = [L_{2n+1} + 1,\ L_{2n+2} - 1],\ \Lambda_{2n+2} = [L_{2n+2},\ L_{2n+3}].$$

(A) For all $n \geq 2$ and $k = 1, \dots, L_{2n} - 1$, we have

$$I_n: \quad \beta(L_{2n+1} + k) = 1000(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}1001,$$

$$K_n: \quad \beta(L_{2n+1} + L_{2n-1} + k) = 1010(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}0001.$$

Moreover, for all $n \geq 2$ and $k = 0, \dots, L_{2n-1}$, we have

$$J_n: \quad \beta(L_{2n+1} + L_{2n-2} + k) = 10010(10)^{-1}\beta(L_{2n-2} + k)(01)^{-1}001001.$$

(B) For all $n \geq 1$ and $k = 0, \dots, L_{2n+1}$ one has

$$\beta(L_{2n+2} + k) = \beta(L_{2n+2}) + \beta(k) = 10\cdots0\,\beta(k)\,0\cdots01.$$

History of RCST

P. Filipponi and E. Hart. The Zeckendorf decomposition of certain Fibonacci-Lucas products. Fibonacci Quart. 36 (1998).

E. Hart, On using patterns in the beta-expansions to study Fibonacci-Lucas products, Fibonacci Quart. 36 (1998).

E. Hart and L. Sanchis, On the occurrence of F_n in the Zeckendorf decomposition of nF_n, Fibonacci Quart. 37 (1999).

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Repaired:

D.: How to add two natural numbers in base phi, Fibonacci Quart. 59 (2021).

More friendly versions:

D.: The structure of base phi expansions, INTEGERS (2024).

What about the $\beta^-(N)$ part?

N	$Z(N)$	$\beta^+(N)$	$\beta^-(N)$
1	1	1	$\cdot$
2	10	10	$\cdot$01
3	100	100	$\cdot$01
4	101	101	$\cdot$01
5	1000	1000	$\cdot$1001
6	1001	1010	$\cdot$0001
7	1010	10000	$\cdot$0001
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The return of Zeckendorf!

THEOREM All Zeckendorf words of even length ending in 1 appear as $\beta^{-}(N)$-blocks.

Zeckendorf word $:=$ word in which 11 does not occur.

Coding the $\beta^{-}(N)$

Let $\Xi_n := \Lambda_{2n-1} \cup \Lambda_{2n} = [L_{2n-1} + 1, L_{2n+1}]$.

The Ξ_n are the intervals where $\beta^{-}(N)$ has length $2n$.

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Surprisingly, on each Ξ_n the $\beta^-(N)$ can be coded by a number $C(\beta^-(N))$ such that all numbers $\{0, 1, \ldots, F_{2n} - 1\}$ appear.

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The code is given by $C(N) = Z^{-1}\big(\beta^-(N)1^{-1}0^{-1}\big)$.

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Here $0's$ as prefix of $\beta^-(N)$ are ignored.
For example: $\beta^-(9)\,1^{-1}0^{-1} = 0101\,1^{-1}0^{-1} = 01$, so $C(9) = 1$.

The structure of the $\beta^-(N)$

THEOREM For all natural numbers n, consider the F_{2n} Zeckendorf words of length $2n$ occurring as $\beta^-(N)$ in the β-expansions of the numbers in Ξ_n. Then these occur in an order given by a permutation Π^{β}_{2n}, which is the orbit of the element $F_{2n} - 1$ under addition by the element F_{2n-2} on the cyclic group $\mathbb{Z}/F_{2n}\mathbb{Z}$.

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EXAMPLE For $n=3$, we have $\Xi_3 = \Lambda_5 \cup \Lambda_6 = \{12, 13, \ldots, 29\}$, furthermore $F_{2n} = F_6 = 8,\ F_{2n}-1 = 7,\ F_{2n-2} = F_4 = 3.$

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On the next slide we see that $\Pi^\beta_6 = (7\,2\,5\,0\,3\,6\,1\,4)$.

$$7 \xrightarrow{+3} 2 \xrightarrow{+3} 5 \xrightarrow{+3} 0 \xrightarrow{+3} 3 \xrightarrow{+3} 6 \xrightarrow{+3} 1 \xrightarrow{+3} 4$$

N	Λ-int.	$\cdot\beta^-(N)$	$\mathrm{C}(N)$
12	Λ_5	·101001	7
13	Λ_5	·001001	2
14	Λ_5	·001001	2
15	Λ_5	·001001	2
16	Λ_5	·100001	5
17	Λ_5	·000001	0
18	Λ_6	·000001	0
19	Λ_6	·000001	0
20	Λ_6	·010001	3
21	Λ_6	·010001	3
22	Λ_6	·010001	3
23	Λ_6	·100101	6
24	Λ_6	·000101	1
25	Λ_6	·000101	1
26	Λ_6	·000101	1
27	Λ_6	·010101	4
28	Λ_6	·010101	4
29	Λ_6	·010101	4

Stop

THE END