Poisson genericity in numeration systems with exponentially mixing probabilities

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This talk is about numeration systems as sequences of symbols of a finite or countable set $\Omega$, called an alphabet.

An overview

Consider a numeration system associated with an invariant exponentially mixing measure.

For almost all infinite sequences of symbols $x$, the number of times that the words $w$ of length $k$ which are in the initial segment of $x$ follows a Poisson law as $k \to \infty$.

Numeration systems covered by our result

- Integer bases and continued fractions.
- Fibred systems with an invariant and exponentially mixing measure (including the Ostrowski continued fraction algorithm in the plane).
- Stochastic processes as aperiodic and irreducible Markov chains.
Motivation

Yuval Peres and Benjamin Weiss proved the result for integer bases $b$

**Poisson for integer bases**

For almost all $x \in [0, 1]$ with respect to the Lebesgue measure, the number of times that words $w$ of length $k$ are in the base $b$ expansion of $x$ follows a Poisson law as $k \to \infty$.

Weiss. Poisson generic points.


Álvarez, Becher and Mereb transcribed their proof and related Poisson genericity with the notion randomness from computability theory.


**Our initial question:** Are the methods of Peres and Weiss amenable to continued fractions?

The symbols in CF expansions are infinitely correlated.
Notations and examples

- \( \Omega \): alphabet finite or countable
- \( x \in \Omega^\mathbb{N} \): infinite sequences of symbols in \( \Omega \)
- \( w \in \Omega^k \): words of length \( k \), for each \( k \geq 1 \)

**Example**

\( \Omega = \{0, \ldots, 9\} \)

\( x = 414213562373095048801688724209698078569671875376948 \ldots \)

\( k = 2 \), \( i = 4 \), \( x[4, 5] = 21 \)

The word \( 69 \) is three times in the first 50 symbols of \( x \).

**Statistics of \( x \) with words \( w \) of two symbols**

<table>
<thead>
<tr>
<th>( j )</th>
<th># Words of two symbols that are ( j ) times in ( x )</th>
<th>Proportion of words of two symbols that are ( j ) times in ( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>61</td>
<td>0.61</td>
</tr>
<tr>
<td>1</td>
<td>30</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>0.08</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.01</td>
</tr>
<tr>
<td>4 or more</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Examples

Decimal expansions, Lebesgue measure
\( \lambda = 1. \)
\( k = 6, \ w \in \Omega^6. \)

**Random \( x \)**

Initial segment of length \( 10^6. \)

In red, the Poisson probability mass function.

In blue/green, the histogram of the proportion of words \( w \) which appears 0, 1, \ldots times

\( x \) is the decimal expansion of a number generated at random

**Champernowne**

\( x \) is the Champernowne number
\( x = 123456789101112131415\ldots \)

The Champernowne number is not Poisson for \( \lambda = 1 \) (Peres and Weiss)
The Poisson distribution

Consider the random allocation of $N$ balls in $K$ bins.

If $N$ is smaller than $K$, a lot of bins will be empty or with exactly one ball, fewer with exactly two, still fewer with exactly three. . . .
The Poisson distribution

Consider $N$ balls and $K$ bins.
The probability $p$ that a bin is allocated is $1/K$.
The expected proportion of bins with exactly $j$ balls, for $j = 0, 1, 2, \ldots$

$$\chi(j) = \binom{N}{j} p^j (1 - p)^{N-j}.$$

When $N$ and $K$ go to infinity but $N/K = \lambda$ is a fixed constant

$$\chi(j) \text{ converges to } e^{-\lambda} \frac{\lambda^j}{j!},$$

the Poisson probability mass function with parameter $\lambda$.

Notation:

$Po(\lambda)$: probability mass function of parameter $\lambda$.

$X \sim Po(\lambda)$: The r.v. $X$ is distributed according to a Poisson of parameter $\lambda$.
Any irrational number in $[0, 1]$ admits a representation of the form:

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} = [a_1, a_2, \ldots] \quad a_i \in \mathbb{N}, \ i \geq 1.$$ 

The natural measure $\mu$ associated with CF is the Gauss measure:

$$d\mu = \frac{dx}{\ln(2)(1 + x)}.$$ 

**Integer bases vs continued fractions**

<table>
<thead>
<tr>
<th>Integer bases</th>
<th>Continued fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite alphabet</td>
<td>Infinite alphabet</td>
</tr>
<tr>
<td>Independence of symbols</td>
<td>infinite correlations between symbols</td>
</tr>
</tbody>
</table>
And beyond

The Ostrowski map

Given irrationals \( x, y \in [0, 1] \) define

\[ T(x, y) = \left( \{1/x\}, \{y/x\} \right), \]

where \( \{t\} := t - \lfloor t \rfloor \) is the fractional part.

\[ \Omega = \{(a, b) \in \mathbb{Z}^2 : a \geq 1, 0 \leq b \leq a\} \]

\( T \) has an invariant and exponentially mixing probability \( \mu \) (Berthé and Lee, 2024)
A general setting: sequences and measures

\( \Omega \): finite or countable set of symbols

\( \mu \): Borel probability measure on \( \Omega^\mathbb{N} \).

\( \mu_k \): the Borel probability measure on \( \Omega^k \) induced by \( \mu \), for each \( k \in \mathbb{N} \).

**Cylinders:** \( w \in \Omega^k \),

\[
C_k(w) = \{ x \in \Omega^\mathbb{N} : x[1,k) = w \}, \quad \mu_k(w) = \mu(C_k(w))
\]

Length of initial segments: \( \lfloor \lambda / \mu_k(w) \rfloor \), \( \mu_k(w) \neq 0 \), for each \( \lambda > 0 \).

Variable of interest: \( M_k(x, w)(\lambda) = \#\{ w \text{ is in } x[1, \lfloor \lambda / \mu_k(w) \rfloor] \} \)

**Example with continued fractions**

\( x = \pi - 3 = [0; 7, 15, 1, 292, 1, 1, 2, 1, 3, 1, 14, 2, 1, \overbrace{1}^{14}, 1, 2, 2, 2, 2, \ldots] \)

\( k = 2, \ w = 12, \)

\[
C_2(12) = \left[ \frac{1}{1+\frac{1}{2}}, \frac{1}{1+\frac{1}{3}} \right] = [2/3, 3/4], \quad \mu_2(12) = \int_{2/3}^{3/4} \mu(x) dx = 0.0708 \ldots
\]

\( \lambda = 1, \ \lambda / \mu_2(12) = 14.2 \ldots \)

\[
M_2(\pi - 3, 12)(1) = 1
\]
Poisson law for point processes on $\mathbb{R}^+$

For $x \in \Omega^N$, $i, k \in \mathbb{N}$ and $w \in \Omega^k$, let the indicator function be

$$I_i(x, w) = 1_{x[i,i+k)=w}$$

For each $k \in \mathbb{N}$, for each $x \in \Omega^N$, on the space $\Omega^k$ with measure $\mu_k$,

$$M^x_k(w)(S) = M_k(x, w)(S) = \sum_{i: i\mu_k(w) \in S} I_i(x, w), \quad \text{for any Borel set } S \subseteq \mathbb{R}^+.$$ 

is a integer-valued random measure on $\mathbb{R}^+$.

The sum runs over $i \in \mathbb{N}$, $i\mu_k(w) \in S$.

Choose $w$ at random and assign to $S$ a nonnegative number

$$S \mapsto M^x_k(w)(S)$$

If $S = (0, \lambda]$, the initial segment is $[\lambda/\mu_k(w)]$ as before.
A point process $X(\cdot)$ on $\mathbb{R}^+$ is an integer-valued random measure on $\mathbb{R}^+$.

A point process $\text{Po}(\cdot)$ on $\mathbb{R}^+$ is Poisson if

- for all disjoint Borel sets $S_1, \ldots, S_m$ included in $\mathbb{R}^+$, the random variables $\text{Po}(S_1), \ldots, \text{Po}(S_m)$ are mutually independent;
- for all bounded Borel sets $S \subseteq \mathbb{R}^+$, the random variable $\text{Po}(S)$ has the distribution of a Poisson random variable with parameter $|S|$, the Lebesgue measure of $S$.

A sequence $X_k(\cdot)_{k \geq 1}$ of point processes converges in distribution to a point process $\text{Po}(\cdot)$ if for every Borel set $S \subseteq \mathbb{R}^+$, the random variables $X_k(S)$ converge in distribution to $\text{Po}(|S|)$ as $k$ goes to infinity.

We write $X_k(\cdot) \xrightarrow{(d)} \text{Po}(\cdot)$

**Fact**

The sequence $(M_k^X(\cdot))_{k \geq 1}$ is a sequence of point processes on $\mathbb{R}^+$.

($\omega$ chosen at random in $\Omega^k$.)
Definition (Poisson genericity)

We say that \( x \in \Omega^N \) is Poisson generic if the sequence \( (M^x_k(\cdot))_{k \geq 1} \) of point processes on \( \mathbb{R}^+ \) converges in distribution to a Poisson point process on \( \mathbb{R}^+ \), as \( k \) goes to infinity.

This means that, for each fixed \( x \), for every Borel set \( S \subseteq \mathbb{R}^+ \),

\[
M^x_k(S) \xrightarrow{(d)} \text{Po}(|S|), \text{ as } k \to \infty.
\]

or, for each \( j \geq 0 \),

\[
\mu_k \left( \{ w \in \Omega^k : M^x_k(w)(S) = j \} \right) \to e^{-|S|} |S|^j / j! \text{ as } k \to \infty.
\]

\(|S|\) is the Lebesgue measure of \( S \).
Assumptions on the probability measure $\mu$ on $\Omega^\mathbb{N}$.

- **Invariant**: $\mu_k(w) = \mu_k(\{x : x[i, i + k) = w\})$ for any $i, k \in \mathbb{N}$ and $w \in \Omega^k$.

- **Exponentially mixing** (nonindependent “ma non troppo”)
  There exists a $0 < \sigma < 1$ such that for any $A, B \subset \Omega^\mathbb{N}$ of positive measure with
  $A$ depending on the first $i$ symbols,
  $B$ depending on the symbols from position $j$, $j \geq i + k$,

  $$\left| \frac{\mu(A \cap B)}{\mu(A)\mu(B)} - 1 \right| = O(\sigma^{j-i-k}).$$

Our main theorem

For any invariant and exponentially mixing probability measure $\mu$ on $\Omega^\mathbb{N}$, $\mu$-almost all $x \in \Omega^\mathbb{N}$ are Poisson generic.
The role of $x$ and $w$ is not symmetric.

For fixed $w \in \Omega^k$, it is feasible to prove the estimate

$$\mathbb{E}_{\mu}[M^w_k(S)] \approx |S| \quad \text{as } k \to \infty$$

for any $S \subset \mathbb{R}^+$ which is a finite union of bounded intervals.

For fixed $x \in \Omega^N$, to obtain estimates of

$$\mathbb{E}_{\mu_k}[M^x_k(S)] \quad \text{as } k \to \infty$$

is not immediate.
Adaptation of Peres and Weiss’ general strategy

**Annealed result. Integrate on** $\Omega^N \times \Omega^k$

- Fix $\omega \in \Omega^k$ and integrate with respect to $\chi \in \Omega^N$. Only finite union of bounded intervals $S$.
  
  Use the Chen-Stein method (only for invariant and exponentially mixing probabilities). Bound the total variation distance between $M^w_k(S)$ and $Po(|S|))$.

- Integrate with respect to $\omega \in \Omega^k$.

- Use Kallenberg’s criterion of convergence for point processes:
  \[ M_k(\cdot) \xrightarrow{(d)} Po(\cdot) \quad \text{as} \ k \to \infty. \]

**Quenched result (almost all} $\chi \in \Omega^N$, integrate on $\Omega^k$)

- With ‘high probability”, for $\chi \in \Omega^N$. Use a concentration result
  \[ M^\chi_k(\cdot) \sim M_k(\cdot) \sim Po(\cdot) \quad \text{as} \ k \to \infty. \]

- From “high probability” to almost all $\chi$: Use Borel Cantelli’s lemma
  Only finite union of bounded intervals $S$.

- Use Kallenberg’s criterion of convergence for point processes and conclude:
  \[ \text{Poisson genericity for almost all} \ \chi \in \Omega^N \]
Sketch of the proof: Annealed

**Annealed result. Integrate on** $\Omega^N \times \Omega^k$

- Fix $w \in \Omega^k$ and integrate with respect to $x \in \Omega^N$. Only finite union of bounded intervals $S$: 
  \[ \mathbb{E}_\mu[M_k^w(S)] \approx |S| \quad \text{and} \quad \mathbb{V}_\mu[M_k^w(S)] \approx |S| + \text{error}(w) \]

  Use the Chen-Stein method: invariant and exponentially mixing probabilities

  *If $X$ is a sum of indicators and its expectation is $\lambda$, the total variation distance between $X$ and $\text{Po}(\lambda)$ is controlled by $|\mathbb{V}[X] - \lambda|$.***

- Integrate with respect to $w \in \Omega^k$: 
  \[ \mathbb{E}_{\mu_k}[\text{error}(w)] \to 0 \quad \text{as} \ k \to \infty. \]

- Use Kallenberg's criterion of convergence for point processes: From finite union of bounded intervals with rationals points to Borel sets. For every Borel set $S$, 
  \[ M_k(S) \xrightarrow{(d)} \text{Po}(|S|) \quad \text{as} \ k \to \infty. \]

  The integration is done with respect to the measure $d\mu \times d\mu_k$. 
Sketch of the proof: Quenched result

Use a concentration result
For some prescribed conditions on $\varphi : \Omega^N \rightarrow \mathbb{R}$, $\varphi$ is to close to its mean:

$$\mu(\{x : |\varphi(x) - E_{\mu}[\varphi]| \geq t\}) \rightarrow 0 \text{ as } t \rightarrow +\infty.$$  

Kontorovich and Ramanan (2007- 2008)
A concentration results holds if $\varphi$ depends on a finite and fixed number of symbols and one of the following conditions holds

- $\Omega$ is finite or
- $\Omega$ is countable and $\varphi$ satisfies the constant weighted Hamming distance property.

Our “functions” $M_k(x, w)(S)$ depend of all symbols of $x$ as $k \rightarrow \infty$.

Our concentration
A concentration result holds for $\varphi : \Omega^N \hookrightarrow \mathbb{R}^+$ if

- $\varphi$ is a “strong” limit of a sequence of functions $(\varphi_N)_{N \geq 1}$, each $\varphi_N$ depends on $N$ symbols.
- Each $\varphi_N$ satisfies a concentration result à la Kontorovich-Ramanan.
Sketch of the proof: Quenched result

**Quenched result (almost all \( x \in \Omega^N \), integrate on \( \Omega^k \))**

- Use a concentration result: with “high probability” on \( x \in \Omega^N \),
  \[
  M^x_k(S) \approx \mathbb{E}_\mu[M_k(S)] \approx \text{Po}(|S|) \quad \text{as } k \to \infty.
  \]

- Use Borel Cantelli’s lemma: from “high probability” to almost all \( x \).
  For almost all \( x \in \Omega^N \), for every finite union of bounded intervals of rationals endpoints \( S \),
  \[
  M^x_k(S) \xrightarrow{\text{(d)}} \text{Po}(|S|) \quad \text{as } k \to \infty.
  \]
  (Integrate with respect to \( \omega \in \Omega^k \))

- Use Kallenberg’s criterion of convergence for point processes.
  From finite union of bounded intervals of rationals end points to Borel sets.
  For almost all \( x \in \Omega^N \),
  \[
  M^x_k(\cdot) \xrightarrow{(d)} \text{Po}(\cdot) \text{ as } k \to \infty.
  \]

**Poisson genericity for almost all \( x \in \Omega^N \)**
Poisson limit law in Dynamical Systems

Return time: the number of visits of a given orbit to a set.

**General goal**

Consider a discrete dynamical system with an invariant mixing probability and a sequence of sets shrinking to a point (satisfying good properties). The distribution of return times is asymptotically Poisson as the measure of the sets goes to zero.

**Early works** (1940–1990): Doeblin-Iosifescu (CF), Pitskell (MC)
**Poisson law of rare events** (1990–): Collet, Coelho, Galves, Hirata, Schmitt.
**Followed by** (2000–): Abadi, Lacroix, Paccaut, Vaienti, Zweimüller, etc.
Some differences with respect to our work

- The role of $\nu$ and $\chi$ is reversed.
- Many works deal with for $S = (0, \lambda)$.
- In many words the exceptional sets (the sets where the limit does not hold) depend on $\lambda$.

Dynamical results:

- error terms,
- periodic orbits (not Poisson).
- different families visited sets (not only cylinders).
- Many different notions of mixing.

Dynamical system methodology
Generating series, transfer operators, Chen-Stein method, dynamical properties of the measure.
Almost all numbers are **decimal Poisson generic** with respect to the Lebesgue measure.

Almost all numbers satisfy Lochs' theorem (1964):
Given \( n \) decimal digits \( d_1, d_2, \ldots, d_n \) of \( x \in [0, 1] \), and \( L_n(x) \) continued fraction digits (partial quotients)

\[
\frac{L_n(x)}{n} \to \frac{6 \ln 10 \ln 2}{\pi^2} \approx 0,97 \quad \text{a.e. } x \quad \text{(Lebesgue measure)}
\]

when \( n \to \infty \).

Let's say that \( x \) is Lochs typical.

**Question:** Is Poisson genericity (normality) Lochs’ invariant?

If a given number \( x \) is **Poisson generic in decimal** and it is **Lochs’ typical**, is it **Poisson generic for continued fractions**?

Lochs theorem for positive entropy numeration systems: Dajani and Fieldsteel (2001)
THANK YOU FOR YOUR ATTENTION
Some bibliography

Poisson point processes for integer bases.


Poisson Law in dynamical systems: two papers with an historical account:

