Substitutive structures on general countable groups

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In this framework, substitutive subshifts are the simplest ones.

Substitutions in 1-D

Thue-Morse substitution (constant-length)

$$\zeta_{TM} : \begin{cases} 0 & \mapsto 01 \\ 1 & \mapsto 10 \end{cases}$$

A fixed point: $x = \ldots 1001.0110 \ldots$

Fibonacci substitution (non constant-length)

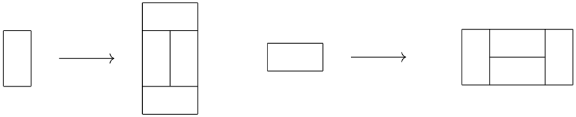
$$\zeta_{F} : \begin{cases} 0 & \mapsto 01 \\ 1 & \mapsto 0 \end{cases}$$

A fixed point: $y = \ldots 01001.01001010 \ldots$

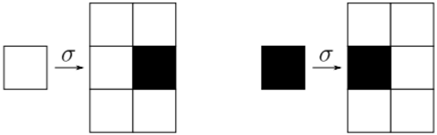
What about beyond the one-dimensional case?

The multidimensional case (square and block substitutions)

The table substitution



Rectangular substitutions



Example given by T. Fernique and V. Lutfalla

Non-linearly recurrent

Constant-shape substitutions (Introduced in 2023 by C.).

- Let $L \in \mathcal{M}(d, \mathbb{Z})$ be an expansion matrix, i.e. L is invertible, $\|L\| > 1$, $\|L^{-1}\| < 1$.
- Let $F \subset \mathbb{Z}^d$ be a fundamental domain of $L(\mathbb{Z}^d)$ in $\mathbb{Z}^d$.
- A constant-shape substitution $\xi \colon \mathcal{A} \to \mathcal{A}^F$

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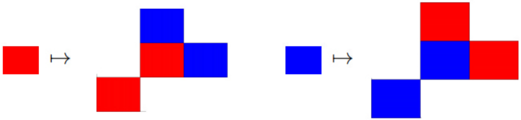
Ex.: $L = 2\mathrm{Id}_{\mathbb{R}^2}$, $F = \{(0,0), (1,0), (0,1), (-1,-1)\}$.



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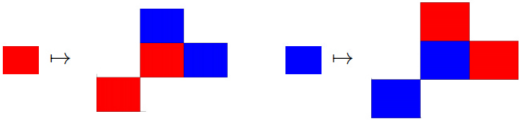

Still a lot of things to do!

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Announcement: We have a definition for $\mathcal{S}$-adic representations on general countable groups! On the rest of the talk we are going to focus on constant-length (or shape) substitutions.

What about general countable groups?

Previous works:

N. Bédaride and A. Hilion (2012): Geometric realizations of two-dimensional substitutive tilings.

S. Beckus, T. Hartnick, F. Pogorzelski (2021): Substitutions on Heisenberg group.

A. Baraviera, R. Leplaideur (2021) and (2023): A strongly aperiodic substitution on $\mathbb{F}_2^+$.

L. Bartholdi, V. Salo (2024): Substitutions on locally finite groups.

A first definition (inspired on the multidimensional case)

- Let G be a countable group.
- Let $\varphi : G \to G$ be an endomorphism (such that $\varphi(G)$ is of finite-index).
- Let F be a set of representatives of right cosets of $G/\varphi(G)$.

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This create a nested sequence of finite-index subgroups

$$G \geq \varphi(G) \geq \varphi^2(G) \geq \dots$$

and a sequence $(F_n)_{n\in\mathbb{N}}$ of set of representatives of right cosets: $F_{n+1} = \varphi(F_n)F_1$

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Remark

We require that $\bigcup_{n\in\mathbb{N}} F_n = G$. *This implies that*

$$\bigcap_{n\geq 0} \varphi^n(G) = \{1_G\}.$$

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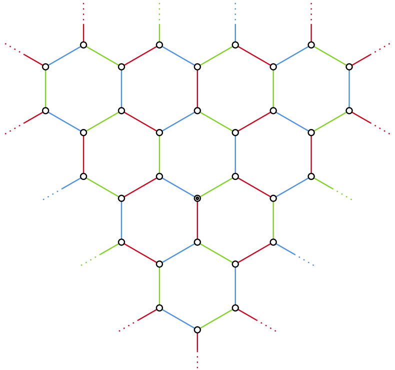
The *Honeycomb Coxeter group*, given by the presentation

$$W = \langle s, t, r \mid s^2 = t^2 = r^2 = (st)^3 = (tr)^2 = (sr)^3 = 1 \rangle.$$



admits the endomorphism defined as $\phi(s) = sts$, $\phi(t) = rsr$ and $\phi(r) = trt$. A set of representatives is $F_\varphi = \{1_W, s, t, r\}$.

A first definition (inspired on the multidimensional case)

The previous one is an example of an expanding endomorphism

Definition

A finitely generated group G *admits an expanding endomorphism* if there exists a finite generating set S, an endomorphism $\varphi : G \to G$ and $\lambda > 1$ such that $[G : \varphi(G)] < +\infty$ and for all $g \in G$

$$d_S(1_G, \varphi(g)) \geq \lambda \cdot d_S(1_G, g).$$

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The *discrete Heisenberg group* of upper triangular 3×3 matrices with 1s in the diagonal, $\mathcal{H}$, given by the presentation

$$\mathcal{H} = \langle x, y, z \mid [x, z], [y, z], [x, y]z^{-1} \rangle,$$

admits an expansive endomorphism: $\phi(x) = x^2$, $\phi(y) = y^2$ and $\phi(z) = z^4$.

A first definition (inspired on the multidimensional case)

As a consequence of multiple results, only finitely generated virtually nilpotent groups admit expanding endomorphisms.

J. Franks (1970): Anosov diffeomorphisms.
D. Farkas (1981): Crystallographic groups and their mathematics.
M. Gromov (1981): Groups of polynomial growth and expanding maps.
G. Gelbrich (1985): Self-similar tilings and expanding homomorphisms.

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(Possible) excluded groups: Free groups.

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List of (possible) excluded groups:

- The Prüfer p-groups.
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Are they excluded to define a substitution?

NO!

A general definition: Tileability

Let G be a countable group.

- If A, B are two subsets of a group G, we say that A is a (left) *monotile* for B if there exists a subset $C \subseteq G$ such that $\{cA\colon c \in C\}$ is partition of B.

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- A sequence of finite sets $(F_n)_{n\in\mathbb{N}}$ is said to be *locally monotileable* if $F_0 = \{1_G\}$ and F_n is a monotile for F_{n+1} for any $n \in \mathbb{N}$.

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- Such sequence is *exhaustive* if $G = \bigcup_{n\in\mathbb{N}} F_n$.

A general definition: Monoform groups

Definition

A countable group G is *monoform with* φ *and* $F_1 \Subset G$ (called support) if $\varphi : G \to G$ is an injective map (called localization map) with $\varphi(1_G) = 1_G$ and $1_G \in F_1$ such that

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The second condition establishes that $|F_n| = |F_1|^n$, hence we do not consider finite groups for our purposes.

Lemma

The Prüfer p-groups, $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *and the free groups are monoform.*

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Proposition

The class of monoform groups is closed under direct product.

Constant-shape substitutions

Let $\mathcal{A}$ be a finite alphabet and G a monoform group with localization map φ and support F_1. A *constant-shape* or *uniform substitution* is a map $\zeta : \mathcal{A} \to \mathcal{A}^{F_1}$.

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Ex.: Thue-Morse in the Prüfer 2-group: Let $G = \mathbb{Z}[1/2]/\mathbb{Z}$ be the Prüfer 2-group, with $\varphi(g) = g/2$ and $F_1 = \{0, 1/2\}$.

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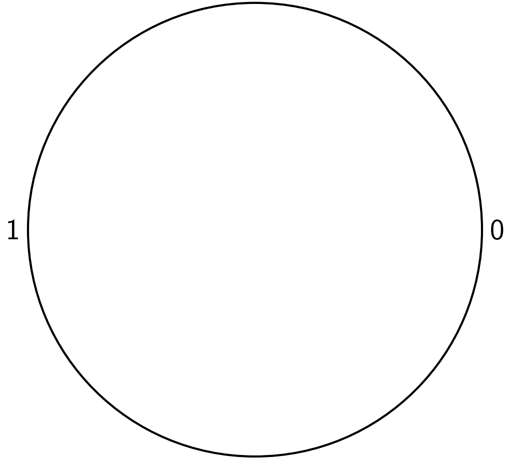
First iteration of the Thue-Morse in the Prüfer 2-group



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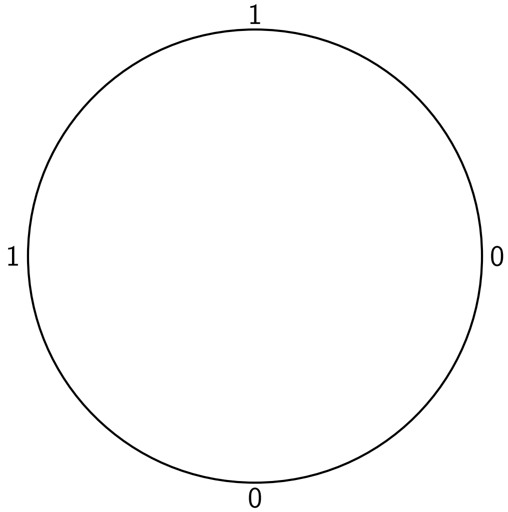
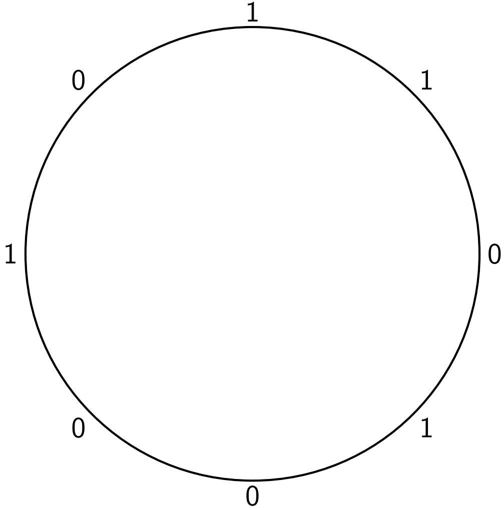
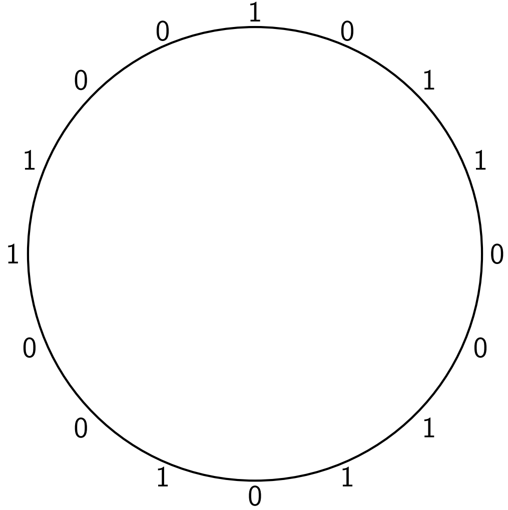


Third iteration of the Thue-Morse in the Prüfer 2-group



Fourth iteration of the Thue-Morse in the Prüfer 2-group



The substitutive subshift associated to ξ is

$$X_\xi = \{x \in \mathcal{A}^G ; \forall F \Subset G, g \in G,\ x|_{gF} \text{ occurs in some } \xi^n(a), n > 0, a \in \mathcal{A}\}$$

with primitivity assumption:

$$\exists n > 0, \quad \forall a, b \in \mathcal{A},\ b \text{ occurs in } \xi^n(a).$$

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Proposition

The substitutive subshift is minimal if and only if the substitution is primitive.

Proposition

If the group is amenable, then an aperiodic primitive substitutive subshift is uniquely ergodic.

Theorem (Bitar, C., Guillon (2024))

If a group G is monoform. There exists a minimal strongly aperiodic G-substitutive subshift (X, S, G).

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Moreover, if φ is an endomorphism such that $\varphi(G)$ is a normal finite-index subgroup of G, then $\mathrm{Aut}(X, S, G) \cong \mathrm{Cent}(G)$.

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Moreover, if φ is an endomorphism such that $\varphi(G)$ is a normal finite-index subgroup of G, then $\mathrm{Aut}(X, S, G) \cong \mathrm{Cent}(G)$.

We also have a formalism to define nonconstant-length substitutions and even $\mathcal{S}$-adic representations on countable groups.

THANKS