Optimal Representations of Gaussian and Eisenstein Integers

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Utrecht, Numeration 2024 = 2 \cdot 10^3 + 0 \cdot 10^2 + 2 \cdot 10^1 + 4 \cdot 10^0
Positional number system: base $\beta$ and digit set $D$

- Case 1: $\beta = \imath - 1$, $D = \{0, \pm1, \pm\imath\}$
- Case 2: $\beta = \omega - 1$, $D = \{0, \pm1, \pm\omega, \pm\omega^2\}$, $\omega = \exp \frac{2\pi\imath}{3}$
Positional number system: base $\beta$ and digit set $D$

- **Case 1:** $\beta = \imath - 1$, $D = \{0, \pm 1, \pm \imath\}$
- **Case 2:** $\beta = \omega - 1$, $D = \{0, \pm 1, \pm \omega, \pm \omega^2\}$, $\omega = \exp \frac{2\pi \imath}{3}$

Both these systems $(\beta, D)$ represent all complex numbers:

$$\left\{ \sum_{k<N} d_k\beta^k : N \in \mathbb{Z}, d_k \in D \right\} = \mathbb{C}$$

$$\left\{ \sum_{k=0}^{N-1} d_k\beta^k : N \in \mathbb{N}, d_k \in D \right\} = \left\{ \begin{array}{ll}
\text{Gaussian integers } \mathbb{Z}[\imath] \text{ in Case 1} \\
\text{Eisenstein integers } \mathbb{Z}[\omega] \text{ in Case 2}
\end{array} \right.$$

The equalities are satisfied even with smaller alphabets $A \subset D$:

- $A = \{0, 1\}$ for $\beta = \imath - 1$
- $A = \{0, 1, -\omega\}$ for $\beta = \omega - 1$

But bigger alphabets $D$ have benefits, while staying closed under multiplication, which simplifies multiplication of representations.
Positional number system: base $\beta$ and digit set $\mathcal{D}$

- **Case 1**: $\beta = \iota - 1$, $\mathcal{D} = \{0, \pm 1, \pm \iota\}$
- **Case 2**: $\beta = \omega - 1$, $\mathcal{D} = \{0, \pm 1, \pm \omega, \pm \omega^2\}$, $\omega = \exp \frac{2\pi \iota}{3}$

Both these systems $(\beta, \mathcal{D})$ represent all complex numbers:

$$\left\{ \sum_{k<N} d_k \beta^k : N \in \mathbb{Z}, d_k \in \mathcal{D} \right\} = \mathbb{C}$$

$$\left\{ \sum_{k=0}^{N-1} d_k \beta^k : N \in \mathbb{N}, d_k \in \mathcal{D} \right\} = \left\{ \begin{array}{l} \text{Gaussian integers } \mathbb{Z}[\iota] \text{ in Case 1} \\ \text{Eisenstein integers } \mathbb{Z}[\omega] \text{ in Case 2} \end{array} \right.$$  

The equalities are satisfied even with smaller alphabets $\mathcal{A} \subset \mathcal{D}$:

- $\mathcal{A} = \{0, 1\}$ for $\beta = \iota - 1$
- $\mathcal{A} = \{0, 1, -\omega\}$ for $\beta = \omega - 1$

but **bigger alphabets** $\mathcal{D}$ have benefits, while staying **closed under multiplication**, which simplifies multiplication of representations.
Why to use bigger alphabets than necessary

**Parallel addition** in $(\beta, \mathcal{D})$: constant time, $p$-local function

- minimal digit set size [Frougny, P., S. (2013)]
  - $\beta = i - 1$: no parallel addition if $|\mathcal{D}| < 5$
  - $\beta = \omega - 1$: no parallel addition if $|\mathcal{D}| < 7$

- Both system Case 1 and Case 2 do allow parallel addition [Legerský, S. (2019)]
Why to use bigger alphabets than necessary

Parallel addition in \((\beta, D)\): constant time, \(p\)-local function

- minimal digit set size [Frougny, P., S. (2013)]
  - \(\beta = \nu - 1\): no parallel addition if \(\#D < 5\)
  - \(\beta = \omega - 1\): no parallel addition if \(\#D < 7\)

- Both system Case 1 and Case 2 do allow parallel addition [Legerský, S. (2019)]

Non-adjacent form (NAF): small number of non-zero digits in \((\beta, D)\)-representation \(x = d_{n-1}d_{n-2}\cdots d_1d_0\)

- \(w\)-NAF: any block of length \(w\) of consecutive digits contains at most one non-zero digit
  - Case 1: any Gaussian integer has 3-NAF-representation
  - Case 2: any Eisenstein integer has 2-NAF-representation

- The \(w\)-NAF-representation of \(x\) is unique, and has the minimal Hamming weight among all representations of \(x\), in both Cases 1 and 2 [Heuberger, Krenn (2011)]
Case 1: \( \beta = \nu - 1 \) and \( D = \{0, 1, \overline{1}, \nu, \overline{\nu}\} \)

Examples of NAF vs. general \((\beta, D)\)-representations:

- \( x = 1 \): represented e.g. by strings \( 1, \overline{\nu}, \overline{0}10\overline{0}\overline{1}0\overline{1}0\overline{1}0\overline{1} \)
- \( x = 2 + \nu \): represented e.g. by strings \( 100\overline{\nu}, \nu0, \overline{\nu}1 \)

A representation of \( x \) is called **optimal**, if no other representation of \( x \) has lower Hamming weight.
Case 1: \( \beta = \nu - 1 \) and \( \mathcal{D} = \{0, 1, \overline{1}, \nu, \overline{\nu}\} \)

Examples of NAF vs. general \((\beta, \mathcal{D})\)-representations:

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- \( x = 2 + \nu \): represented e.g. by strings \(100\overline{\nu}, \nu0\overline{\nu}, \nu1\)

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Our goal:

- find a formula to express

\[
f(x) = \text{number of optimal representations of } x \in \mathbb{Z}[\beta]
\]
Case 1: \( \beta = \varpi - 1 \) and \( D = \{0, 1, \overline{1}, \varpi, \overline{\varpi}\} \)

Examples of NAF vs. general \((\beta, D)\)-representations:

- \( x = 1 \): represented e.g. by strings \(1, \overline{\varpi}, \overline{\varpi}010\overline{1}0\overline{1}01\overline{\varpi}0\overline{1}\)
- \( x = 2 + \varpi \): represented e.g. by strings \(100\overline{\varpi}, \varpi0, \varpi1\)

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**Our goal:**

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\[
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\]

- for given \( N \in \mathbb{N} \):
  - describe numbers \( x \in \mathbb{Z}[\beta] \) having \( w \)-NAF-representation with Hamming weight \( \leq N \) and with the **maximal value** \( f(x) \)
Case 1: \( \beta = \iota - 1 \) and \( D = \{0, 1, \overline{1}, \iota, \overline{\iota}\} \)

Examples of NAF vs. general \((\beta, D)\)-representations:

- \( x = 1 \): represented e.g. by strings \( 1, \overline{\iota}, \iota 010\overline{\iota}010\iota0\iota0\overline{\iota} \)
- \( x = 2 + \iota \): represented e.g. by strings \( 100\overline{\iota}, \iota 0\iota, \overline{\iota}1 \)

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  - describe numbers \( x \in \mathbb{Z}[\beta] \) having \( w \)-NAF-representation with Hamming weight \( \leq N \) and with the **maximal value** \( f(x) \)
  - determine the **average value of** \( f(x) \) on the set

\[ \mathcal{M}_N = \{ x \in \mathbb{Z}[\beta] : \text{length of } w \text{-NAF-representation of } x \text{ is } \leq N \} \]
Case 0: $\beta = 2$ and $\mathcal{D} = \{0, 1, \overline{1}\}$

System deeply explored, and questions already responded earlier:

Each $x \in \mathbb{Z}$ has unique 2-NAF-representation and its Hamming weight is minimal among all $(\beta, \mathcal{D})$-representations of $x$ [Reitwiesner (1960)]
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[Reitwiesner (1960)]

Known results about maximal and average values of \( f(x) \)
[Grabner, Heuberger (2006)]:

- If 2-NAF-representation of \( x \in \mathbb{Z} \) has at most \( N \) non-zero digits, then \( f(x) \leq F_{N+1} \), where \((F_N) = \) Fibonacci sequence.
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- Let \( \lambda \in (2, 3) \) be a root of \( t^3 - t^2 - 3t + 1 \). There exists a constant \( d > 0 \) such that the average value of \( f(x) \) equals

\[
\frac{1}{\#\mathcal{M}_{N}} \sum_{x \in \mathcal{M}_{N}} f(x) = d \left( \frac{\lambda}{2} \right)^N (1 + o(1)),
\]

with \( \frac{\lambda}{2} \sim 1.08504 \).
Case 1: \( \beta = \iota - 1 \) and \( \mathcal{D} = \{0, 1, \overline{1}, \iota, \overline{\iota}\} \)

Step 1: **Transducer** on 21 vertices
- acts on input length = output length of 1 or 3 digits
- transforms general \( \mapsto \) 3-NAF representation of \( x \) in \((\beta, \mathcal{D})\)
- outputs on edges: 0, 001, 00\overline{1}, 00\iota, 00\overline{\iota}
Case 1: $\beta = \bar{i} - 1$ and $\mathcal{D} = \{0, 1, \bar{1}, i, \bar{i}\}$

Step 1: **Transducer** on 21 vertices
- acts on input length = output length of 1 or 3 digits
- transforms general $\mapsto$ 3-NAF representation of $x$ in $(\beta, \mathcal{D})$
- outputs on edges: $0, 001, 00\bar{1}, 00i, 00\bar{i}$

Step 2: Limiting the transducer only to **optimal** representations gives **graph** $G$ on 9 vertices
- each oriented path in $G$ starting and ending in 0 and respecting colours of the 3-NAF representation of $x$ provides **optimal** representation of $x$
- 1-1 correspondence
Case 1: \( \beta = i - 1 \) and \( \mathcal{D} = \{0, 1, i, i\} \)

Step 1: **Transducer** on 21 vertices
- acts on input length = output length of 1 or 3 digits
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- outputs on edges: \(0\), \(00\), \(001\), \(00\), \(00\), \(00i\), \(00\overline{i}\)

Step 2: Limiting the transducer only to **optimal** representations gives graph \( G \) on 9 vertices
- each oriented path in \( G \) starting and ending in 0 and respecting **colours** of the **3-NAF** representation of \( x \) provides **optimal** representation of \( x \)
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Case 1: Example - optimal paths in $G$ for $x = 2 + i$

$x = 2 + i = \begin{bmatrix} 001 \\ 00i \end{bmatrix}$

has 3 optimal $(\beta, D)$-representations:

- $2 + i = 100i$
- $2 + i = i0i$
- $2 + i = i1$

$\rightarrow$ 3 paths in $G$ from 0 to 0 coloured with (red)(yellow)
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- $2 + i = i0i$
- $2 + i = \bar{i}1$

→ 3 paths in $G$ from 0 to 0 coloured with (red)(yellow)
Case 1: Number of paths in graph $G$

Step 3: **Incidence matrix** $A_c \in \mathbb{N}^{9 \times 9}$ for each colour in graph $G$:

$$c \in C := \{[0], [001], [001], [00\hat{1}], [00\hat{1}]} \rightarrow A_c$$

$$A_{001} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$(A_c)_{q',q} = \text{number of edges from } q \text{ to } q' \text{ in } G \text{ with colour } c$$
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$(A_c)_{q', q} = \text{number of edges from } q \text{ to } q' \text{ in } G$ with colour $c$

If $u_1 u_2 u_3 \cdots u_n \in C^*$ is the 3-NAF representation of $x \in \mathbb{Z}[\overline{i}]$, then the number of optimal representations of $x$ is

$$f(x) = eA_{u_1}A_{u_2} \cdots A_{u_n} e^\top,$$

where $e = (1, 0, 0, \ldots, 0) \in \mathbb{Z}^9$

Function $f : C^* \mapsto \mathbb{N}$ is a **recognizable series**, in terminology of [Berstel, Reutenauer (2013)]
Case 1: \( \beta = \nu - 1 \) and \( \mathcal{D} = \{0, 1, \bar{1}, \nu, \bar{\nu}\} \)

Step 4: Finding the maximal number of optimal representations:
\[
\max \{ e A_{u_1} A_{u_2} \cdots A_{u_n} e^\top : \text{Hamming weight of } u_1 u_2 \cdots u_n \text{ is } \leq N \}
\]
Case 1: $\beta = \iota - 1$ and $\mathcal{D} = \{0, 1, \iota, \bar{\iota}, \iota\}$

Step 4: Finding the maximal number of optimal representations:
\[
\max \{ eA_{u_1}A_{u_2}\cdots A_{u_n}e^\top : \text{Hamming weight of } u_1u_2\cdots u_n \text{ is } \leq N \}
\]

Theorem
Let $(s_N)_{N \geq 0}$ be the sequence with $s_0 = 1$, $s_1 = \frac{3}{2}$, $s_2 = 3$, and
\[
s_{N+3} = s_{N+2} + 2s_{N+1} + 2s_N \quad \text{for every } N \in \mathbb{N}.
\]

- If the 3-NAF representation of $x$ contains at most $N$ non-zero digits, $N \geq 2$, then the number of optimal representations $f(x) \leq s_N$.

- There exist only 8 Gaussian integers $x \in \mathbb{Z}[\iota]$, $x \notin \beta\mathbb{Z}[\iota]$ whose 3-NAF representation contains $N$ non-zero digits, $N \geq 2$, and which have $s_N$ optimal representations.
Case 1: \( \beta = \iota - 1 \) and \( \mathcal{D} = \{0, 1, \overline{1}, \iota, \overline{\iota}\} \)

\[ \mathcal{M}_N = \{x \in \mathbb{Z}[\iota] : \text{length of 3-NAF representation of } x \text{ is } \leq N\}. \]

Theorem

Let \( \lambda \in (2, 3) \) be a root of \( t^7 - t^6 - 8t^4 + 3t^3 + t^2 - 2t + 2 \). Then the average number of optimal representations of Gaussian integers in \( \mathcal{M}_N \) equals

\[
\frac{1}{\#\mathcal{M}(N)} \sum_{x \in \mathcal{M}(N)} f(x) = \left( \frac{\lambda}{2} \right)^N (d + o(1)) ,
\]

with \( \lambda/2 \sim 1,13674 \) and \( d \sim 0,76257 \).
Case 2: \( \beta = \omega - 1 \) and \( \mathcal{D} = \{0, \pm 1, \pm \omega, \pm \omega^2\} \)

2-NAF representations of Eisenstein integers \( \mathbb{Z}[\omega] \), \( \omega = \exp \frac{2\pi i}{3} \)

- 2-NAF using 7 digit blocks from set \( \{0, 01, 0\omega, 0\omega^2, 0\bar{1}, 0\bar{\omega}, 0\bar{\omega}^2\} \) represented by 7 colours in graph \( G \)
- incidence matrix \( A_c \in \mathbb{N}^{7 \times 7} \) for each colour in \( G \)
Case 2: \( \beta = \omega - 1 \) and \( \mathcal{D} = \{0, \pm 1, \pm \omega, \pm \omega^2\} \)

2-NAF representations of Eisenstein integers \( \mathbb{Z} [\omega] \), \( \omega = \exp \frac{2\pi i}{3} \)

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Theorem

Let \( s_N = \left\lfloor \frac{1}{7} \left( 6 \cdot 2^N + 1 \right) \right\rfloor \), \( N \in \mathbb{N} \) be the sequence given by

\[
\begin{align*}
s_0 &= 1, \\ s_1 &= 1, \\ s_2 &= 3, \\ s_{N+3} &= s_{N+2} + s_{N+1} + 2s_N + 1.
\end{align*}
\]

- 2-NAF representation of \( x \) with at most \( N \) non-zero digits has the number of optimal representations \( f(x) \leq s_N \).

- For \( N \geq 4 \), there are only 12 Eisenstein integers \( x \in \mathbb{Z} [\omega] \), \( x \notin \beta \mathbb{Z} [\omega] \) with 2-NAF representation containing \( N \) non-zero digits, and having \( s_N \) optimal representations.
Case 2: \( \beta = \omega - 1 \) and \( \mathcal{D} = \{0, \pm 1, \pm \omega, \pm \omega^2\} \)

\[ \mathcal{M}_N = \{x \in \mathbb{Z}[\omega] : \text{length of 2-NAF representation of } x \text{ is } \leq N\}. \]

**Theorem**

Let \( \lambda \) be the bigger root of \( t^2 - 3t - 2 \). Then the average number of optimal representations of Eisenstein integers in \( \mathcal{M}_N \) equals

\[
\frac{1}{\# \mathcal{M}(N)} \sum_{x \in \mathcal{M}(N)} f(x) = \left(\frac{\lambda}{3}\right)^N (d + o(1)),
\]

with \( \frac{\lambda}{3} \sim 1.18672 \) and \( d \sim 0.78144 \).
Thanks for your attention