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Low discrepancy words and dynamical systems

V. Berthé, O. Carton, N. Chevallier, W. Steiner, R. Yassawi

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The chairperson assignment problem

- We are given k states which form a union.
- Every year a union chairperson has to be selected.
- At any time the accumulated number of chairpersons from each state has to be proportional to its weight.

How to get in an effective way a fair assignment?

From assignments to symbolic discrepancy

Take a sequence $u = (u_n)_n \in \{1, \cdots, d\}^{\mathbb{N}}$

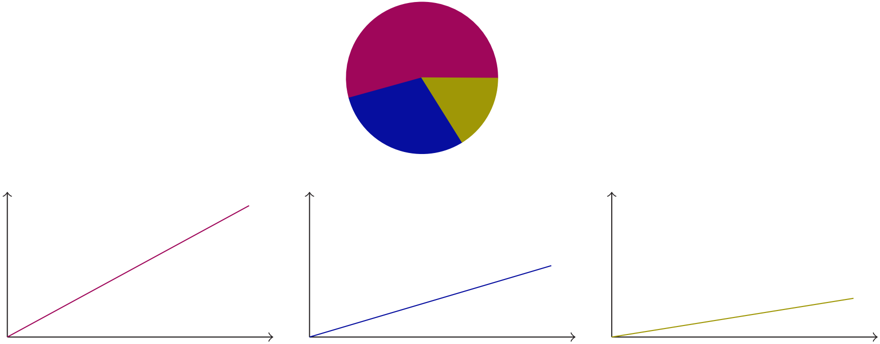
The frequency α_a of the letter a in u is defined as the following limit, if it exists

$$\alpha_a = \lim_{n \to \infty} \frac{1}{n} \mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\}$$

The discrepancy of $u = (u_n)_n$ is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_a \sup_{n \in \mathbb{N}} \left| \mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\} - n\alpha_a \right|$$

Discrepancy



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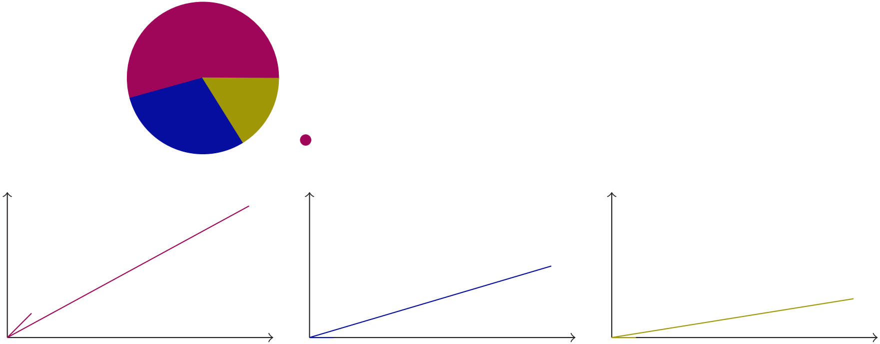

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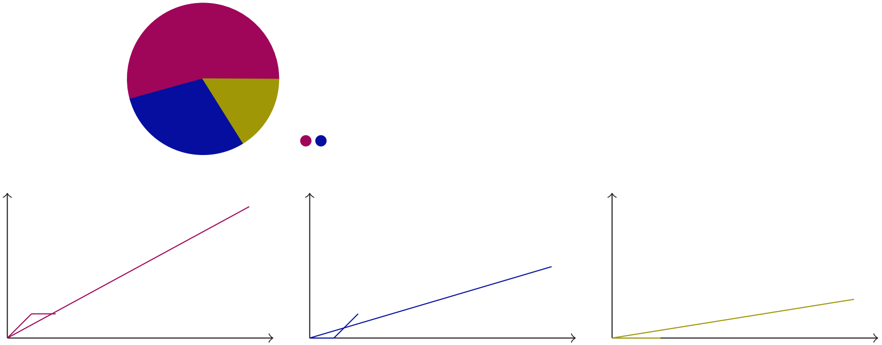

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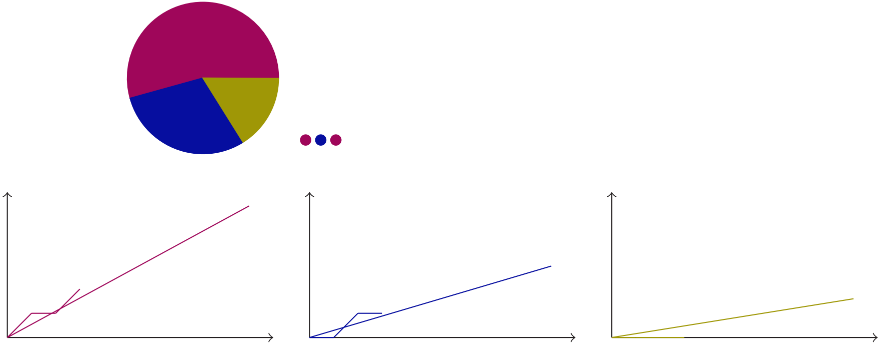

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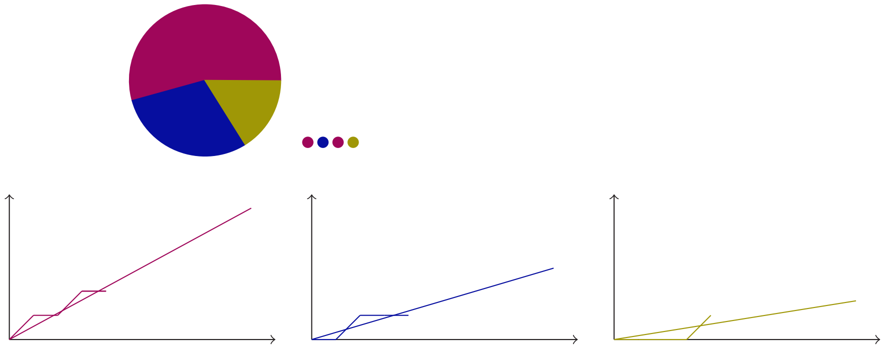

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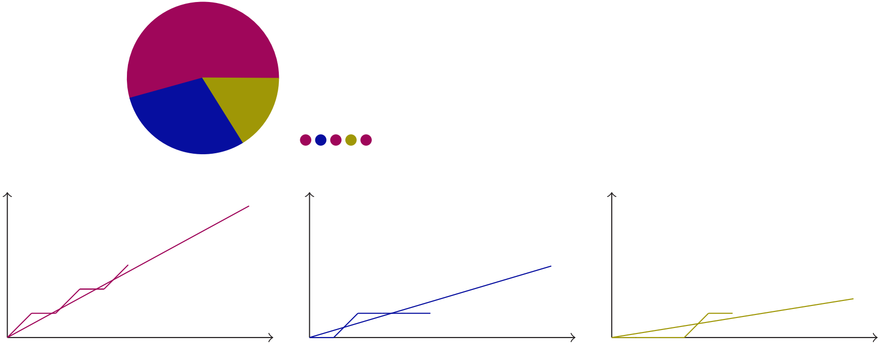

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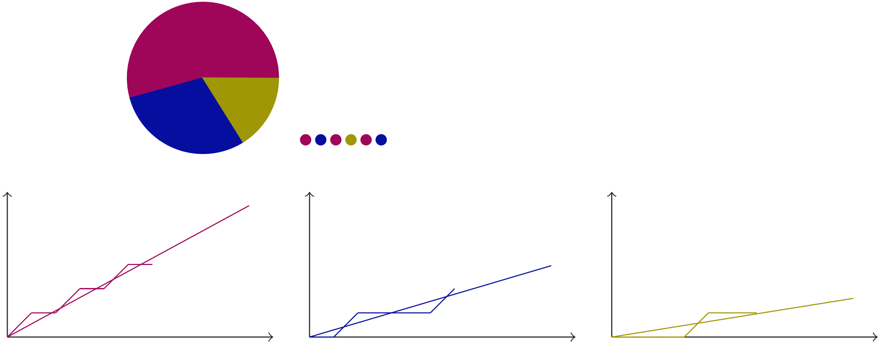

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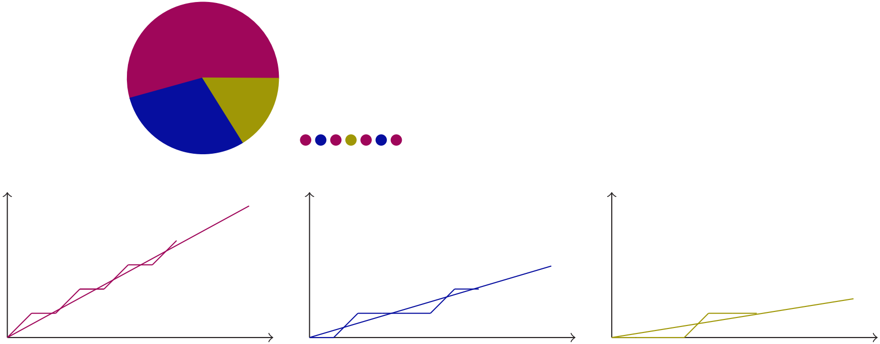

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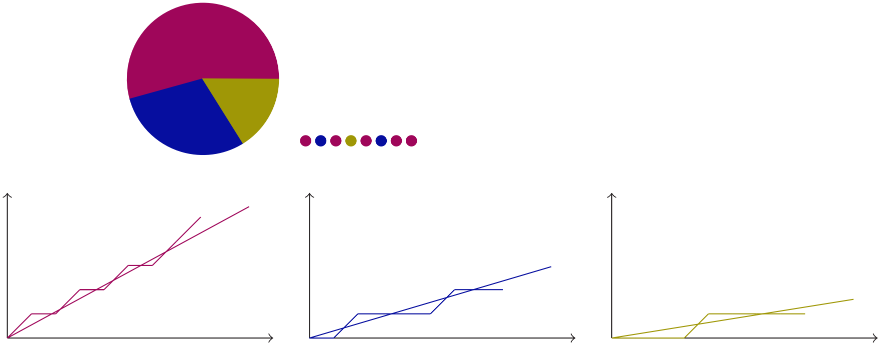

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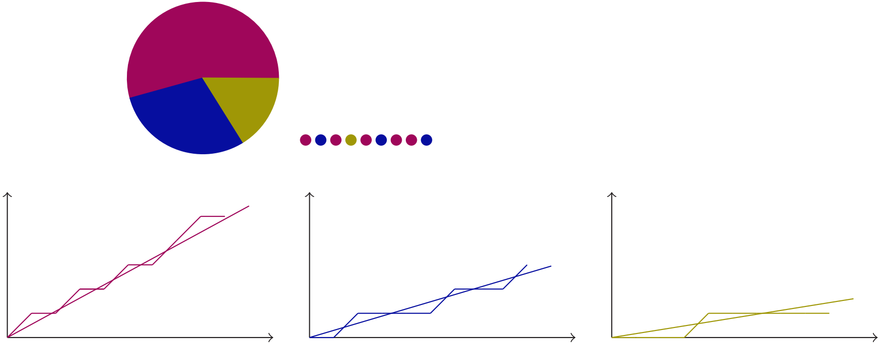

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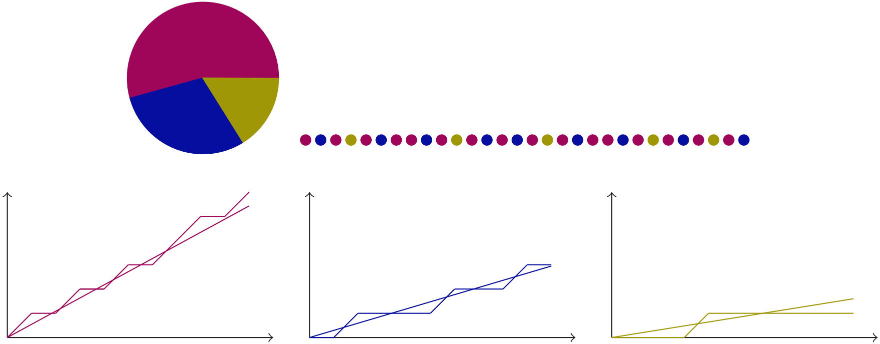

How small can the discrepancy be?

Let $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$ be a frequency vector for the letters

The discrepancy of $u = (u_n)_n$ is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_a \sup_{n \in \mathbb{N}} |\mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\} - n\alpha_a|$$

Theorem [Niederreiter, Meijer, Tijdeman] One has

$$D_d := \sup_{\boldsymbol{\alpha}} \inf_u \Delta_{\boldsymbol{\alpha}}(u) = 1 - \frac{1}{2d-2}$$

Outline

- R. Tijdeman has given an algorithmic way to construct fairly distributed sequences u with $\Delta_{\boldsymbol{\alpha}}(u) \leq 1 - \frac{1}{2d-2}$
- When $d = 2$, $D_2 = 1/2 \rightsquigarrow$ Sturmian sequences
- We revisit Tijdeman's construction in dynamical terms
- We provide constructions of fairly distributed sequences

The two-letter case

The sequences having the smallest discrepancy on a two-letter alphabet are Sturmian sequences.

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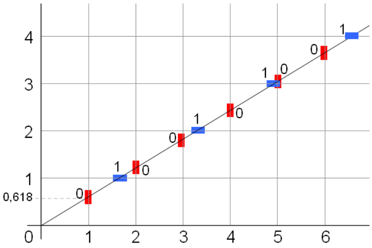
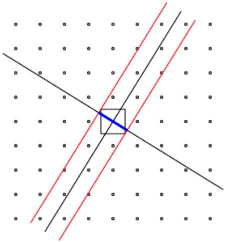
Sturmian sequences are codings of discrete lines.





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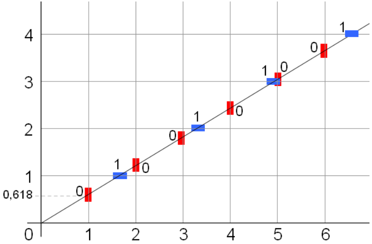

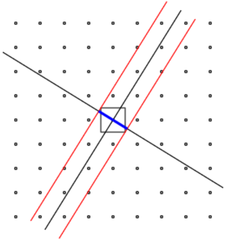

A trajectory for a discrete-time dynamical system

We consider orbits/trajectories of points of X under the action of the map $T : X \to X$

$$\{T^n x \mid n \in \mathbb{N}\}$$

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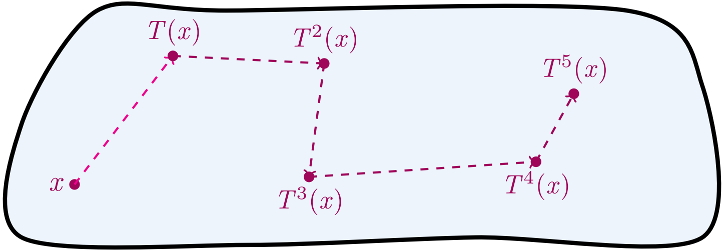

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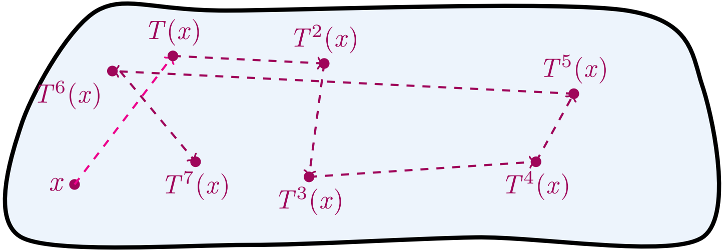
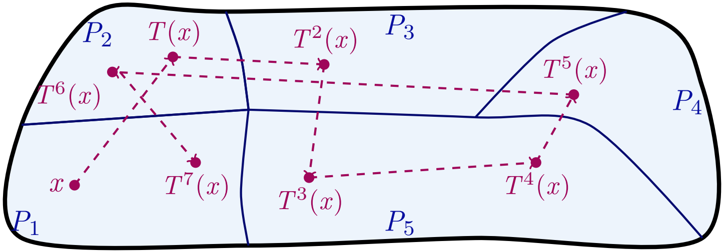


And a coding of a trajectory



The coding works as follows

$$u_n = i \text{ if and only if } T^n(x) \in P_i$$

$$u = (u_n)_n = 12355421 \cdots$$

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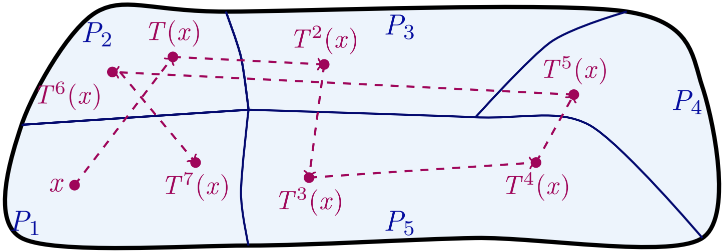

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$$x \mapsto Tx, \;\; u \mapsto 2355421 \cdots$$

Symbolic dynamics

- The shift T acts on $\mathcal{A}^{\mathbb{Z}}$ as $T((u_n)_n) = (u_{n+1})_n$
- A subshift (X, T) is a closed shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$
- Cylinders $[v] = \{u \in X,\ u_0 \cdots u_{|v|-1} = v\} \rightsquigarrow$ Intervals
- Factors/Subwords

$$u = abaababaab \underbrace{aa} babaababaab \cdots$$

aa is a factor, bb is not a factor

- The factor complexity $p_X(n)$ counts the number of factors of length n

Symbolic models for circle rotations

The sequences having the smallest discrepancy on a two-letter alphabet are
Sturmian sequences

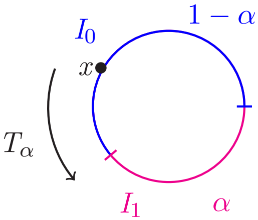
Consider the translation $(\mathbb{T}, T_\alpha)$ where $T_\alpha \colon x \mapsto x + \alpha \bmod 1$ and the coding map

$$\nu \colon [0,1) \to \{0,1\}, \quad \nu(x) = 0 \quad \text{if } x \in I_0, \quad \nu(x) = 1 \quad \text{if } x \in I_1$$

where

$$I_0 = [0, 1-\alpha), \; I_1 = [1-\alpha, 1)$$

The trajectory of x for T_α is coded by $u \in \{0,1\}^{\mathbb{Z}}$ with $u_n = \nu(T_\alpha^n(x))$ for all n



A natural measure of order: factor complexity

What kind of information can the dynamical viewpoint offer here?

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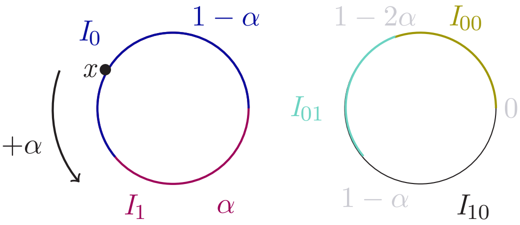
$$u = 0100101001001010010100 1\cdots$$

Does the word 00 occur in the sequence? Does it have a frequency? Does it have bounded discrepancy?



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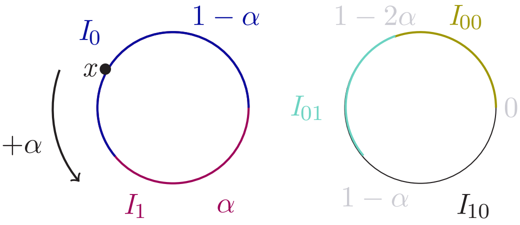
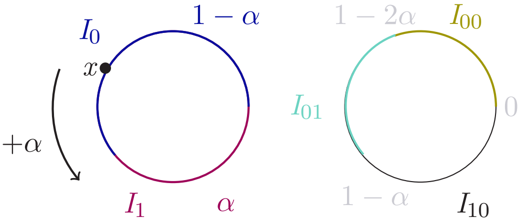


The factors of length n of u are in one-to-one correspondence with the $n+1$ intervals of $\mathbb{T}$ whose end-points are given by

$$-k\alpha \bmod 1 \qquad \text{for } 0 \le k \le n$$

By uniform distribution of $(k\alpha)_k$ modulo 1, the frequency of a factor w of a Sturmian sequence is equal to the length of I_w

Bounded remainder sets



Bounded remainder set A measurable set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \leq n \leq N; T_\alpha^n(0) \in X\} - N\mu(X)| \leq C$$

[Kesten'66] Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$

Letters and even all the factors of Sturmian sequences have bounded discrepancy

Discrepancy for Kronecker sequences

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d$ with $1, \alpha_1, \cdots, \alpha_d$ $\mathbb{Q}$-linearly independent. Consider the Kronecker sequence in $[0,1]^d$

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\})_n$$

associated with the translation over $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$$T_{\boldsymbol{\alpha}} \colon \mathbb{T}^d \to \mathbb{T}^d, \ x \mapsto x + \boldsymbol{\alpha}$$

One has

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\}) = T_{\boldsymbol{\alpha}}^n(0)$$

Discrepancy for Kronecker sequences

Consider the minimal translation over $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$$T_{\boldsymbol{\alpha}} \colon \mathbb{T}^d \to \mathbb{T}^d,\ x \mapsto x + \boldsymbol{\alpha} \mod 1, \quad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$$

Discrepancy Global property

$$\Delta_N(\boldsymbol{\alpha}) = \sup_{B \text{ box}} \left|\mathrm{Card}\,\{0 \leq n < N;\, T_{\boldsymbol{\alpha}}^n(0) \in B\} - N \cdot \mu(B)\right|$$

[Khintchine, Beck] $\Delta_N(\boldsymbol{\alpha})$ is a.e. between

$$(\log N)^d \log\log N \quad \text{and} \quad (\log N)^d (\log\log N)^{1+\varepsilon}$$

Bounded remainder set Local property A measurable set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \leq n \leq N;\, T_{\boldsymbol{\alpha}}^n(0) \in X\} - N\mu(X)| \leq C$$

Bounded remainder sets for toral translations

Bounded remainder set A measurable set X for which there exists $C > 0$ s.t. for all N

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[Kesten'66] $d = 1$ Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$

[Grepstad-Lev'15, Haynes-Kelly-Koivusalo'17] Any parallelotope in $\mathbb{R}^d$ spanned by vectors $v_1, \cdots, v_d$ belonging to $\mathbb{Z}\boldsymbol{\alpha} + \mathbb{Z}^d$ is a bounded remainder set for the minimal translation $T_{\boldsymbol{\alpha}}$

$$T_{\boldsymbol{\alpha}} \colon \mathbb{T}^d \to \mathbb{T}^d,\ x \mapsto x + \boldsymbol{\alpha} \mod 1, \quad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$$

The ubiquitous Fibonacci word

Take the golden ratio $\alpha = \frac{\sqrt{5}+1}{2}$ and the dynamical system

$$x \mapsto x + \alpha \text{ modulo } 1$$

$$\alpha^2 = \alpha + 1 \rightsquigarrow \text{self-similarity}$$

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The Fibonacci substitution

$$\sigma(u) = u \text{ with } \sigma : 0 \mapsto 01,\ 1 \mapsto 0$$

$$u = \sigma^{\omega}(1) = 010010100100101 \cdots$$

Theorem The symbolic dynamical system (X_σ, T) is isomorphic to the geometric dynamical system $(\mathbb{T}, T_{\frac{1+\sqrt{5}}{2}})$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

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Zeckendorf numeration

$$n = \sum_{i=1}^{k} \varepsilon F_i, \varepsilon_i \in \{0,1\},\ 11\ \nexists$$

Fair assignments in general dimension

The best assignments for $d = 2$ code the simplest (discrete-time) dynamical systems.

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And now for $d \geq 3$?

- Given a frequency vector $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \in [0,1]^d$ such that $\sum_{i=1}^d \alpha_i = 1$, R.Tijdeman ('80) has given an algorithmic way to construct a sequence u with $\Delta_{\boldsymbol{\alpha}}(u) \leq 1 - \frac{1}{2d-2}$.

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Theorem [B.-Carton-Chevallier-Steiner-Yassawi] Let u be a Tijdeman sequence with a frequency vector $\boldsymbol{\alpha}$ which has rationally independent coordinates. Then, the sequence u has factor complexity of order n^{d-1}.

The sequence u is a symbolic coding of a translation $T_{\boldsymbol{\alpha}}$ via a partition of a fundamental domain of $\mathbb{T}^{d-1}$ into d finite unions of polytopes such that $T_{\boldsymbol{\alpha}}$ is a translation by a vector on each of the polytopes.

Dynamical systems and Tijdeman sequences

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Consider the minimal translation $T_{\boldsymbol{\alpha}}$

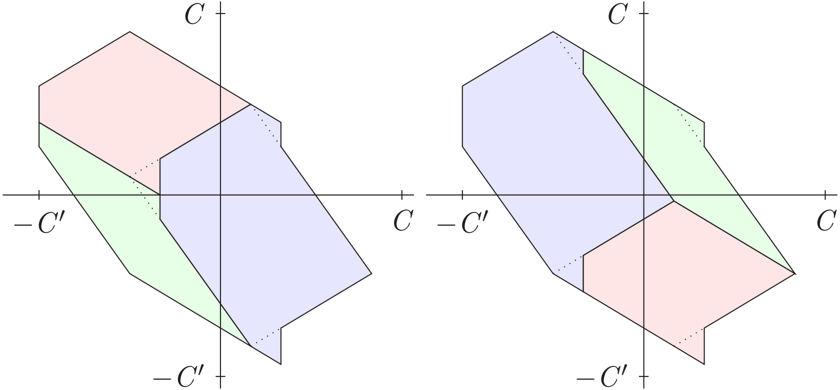
$$T_{\boldsymbol{\alpha}}: \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}, \quad \mathbf{x} \mapsto \mathbf{x} + (\alpha_1, \ldots, \alpha_{d-1}) \pmod{\mathbb{Z}^{d-1}}.$$

Theorem [B.-Carton-Chevallier-Steiner-Yassawi] Let u be a Tijdeman sequence with $\boldsymbol{\alpha} = (\alpha_i)_{1 \leq i \leq d}$ having rationally independent coordinates.

- The sequence u has factor complexity of order n^{d-1}.
- The sequence u is a symbolic coding of $T_{\boldsymbol{\alpha}}$ via a partition of a fundamental domain of $\mathbb{T}^{d-1}$ into d finite unions of polytopes such that $T_{\boldsymbol{\alpha}}$ is a translation by a vector on each of the polytopes.
- The sequence u generates a minimal and uniquely ergodic subshift which has discrete spectrum.

A fundamental domain by polygons

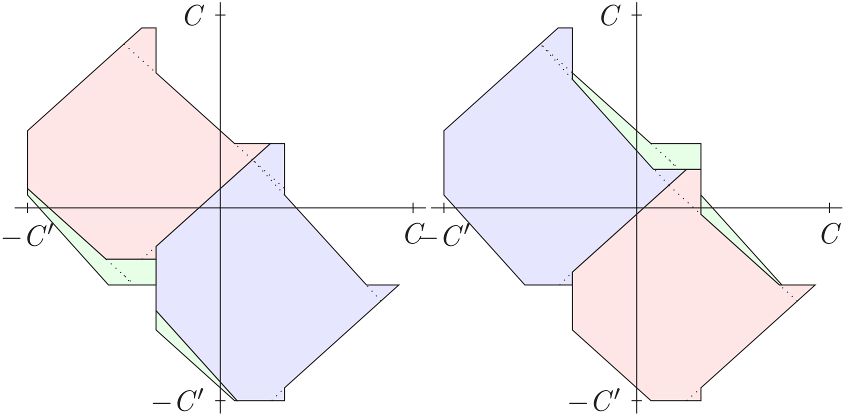
Take $d = 3$, $\boldsymbol{\alpha} \approx (.5, .3, .2)$, $C = C' = 3/4$



Tijdeman sequences code orbits of the corresponding exchange of domains.

This yields a factor complexity of order $n^{d-1} = n^2$

Take $d = 3$, $\boldsymbol{\alpha} \approx (.5, .45, .05)$, $C = C' = 3/4$. The atoms of the partition are unions of polygons.



What does “order” mean for subshifts?

A subshift (X, T) with $X \subset \mathcal{A}^{\mathbb{Z}}$ is simple if

- it has few factors $p_X(n) \leq Cn$ for all n
- it has bounded discrepancy for letters and factors
- it codes a simple dynamical system (a group translation)

What are the relations between these notions of order?

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What are the relations between these notions of order?

Theorem [D. Creutz, R. Pavlov] If $\limsup p_X(n)/n < 3/2$, then X has measurably isomorphic to a group translation

Fairly distributed shifts

How to construct minimal shifts X over the alphabet $\{1, 2, \cdots, d\}$ satisfying the following conditions

- the letter frequencies in X are given by $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$
- X has bounded discrepancy for all its factors
- X has linear factor complexity

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- X has bounded discrepancy for all its factors
- X has linear factor complexity
- X is a symbolic coding of a toral translation

Let us start from the dynamical system given by the translation

$$T_{\boldsymbol{\alpha}} : \mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\alpha} \mod 1$$

How to find a good partition?

How to produce symbolic codings for translations

How to produce fair assignments/ fairly distributed sequences/symbolic codings of $T_{\boldsymbol{\alpha}}$ for a given vector of letter frequencies $\boldsymbol{\alpha}$?

- We apply a multidimensional continued fraction algorithm that generates nonnegative matrices

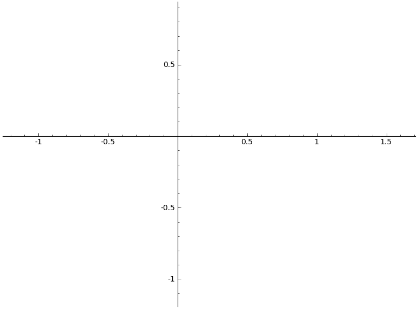
$$\boldsymbol{\alpha} \mapsto (M_n)_n \text{ with } \quad \boldsymbol{\alpha} \in \bigcap_n M_1 \cdots M_n \mathbb{R}_+^d$$

- that generates in turn a sequence of substitutions $\boldsymbol{\alpha} \mapsto (M_n)_n \mapsto \boldsymbol{\sigma} = (\sigma_n)_n$
- and thus sequences $u = \lim \sigma_0 \cdots \sigma_n(a) \rightsquigarrow (X_{\boldsymbol{\sigma}}, T)$ (S-adic formalism)

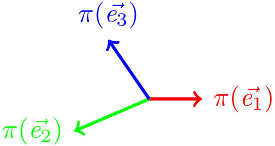
Rauzy fractal and the Tribonacci substitution

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1 \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\sigma^{\omega}(1) = 1213121121312121312112 13\cdots$$



π projection along the expanding eigenline onto the contracting plane of the incidence matrix of M_σ



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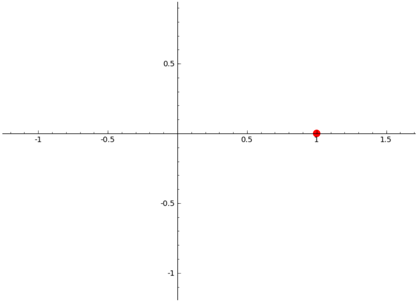

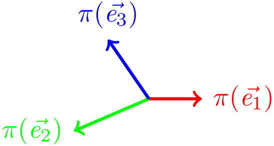

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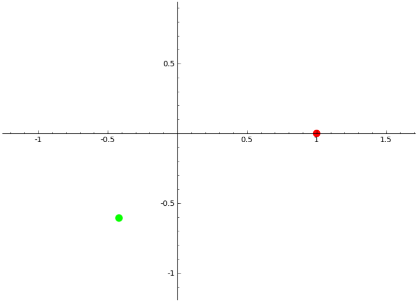

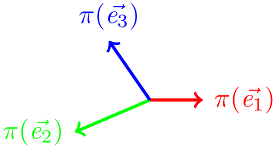

Rauzy fractal and the Tribonacci substitution

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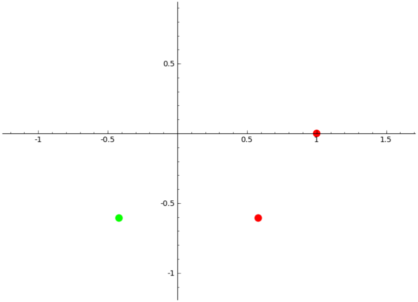

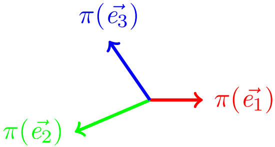

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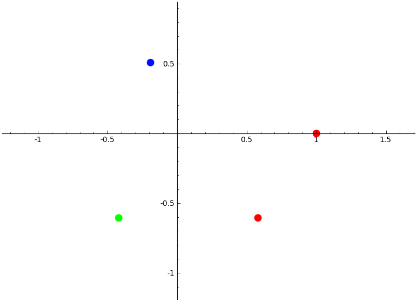

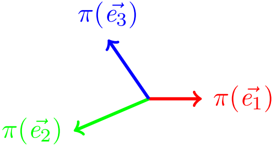

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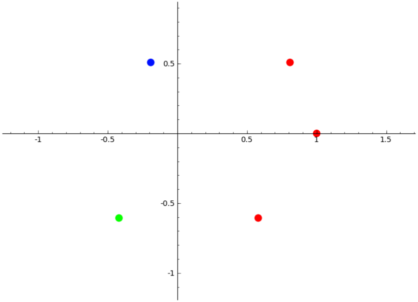

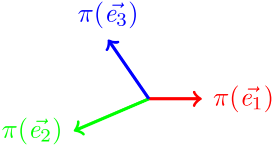

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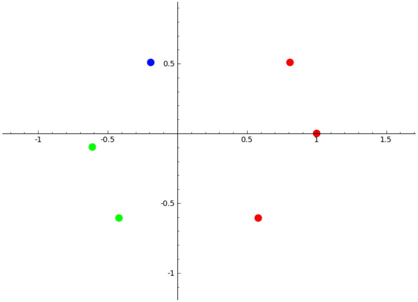

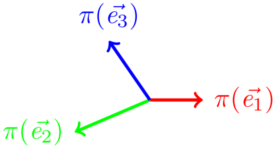

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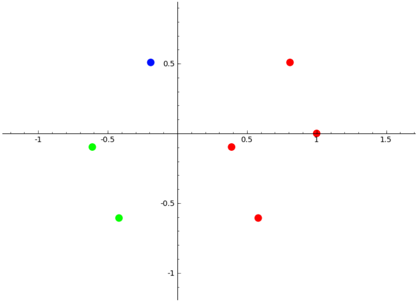

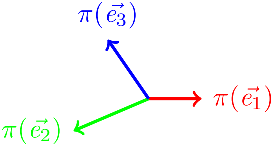

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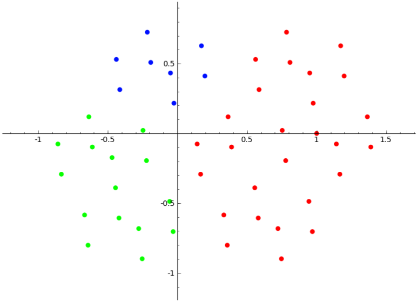

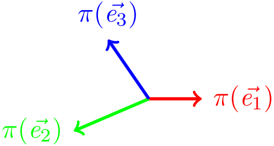

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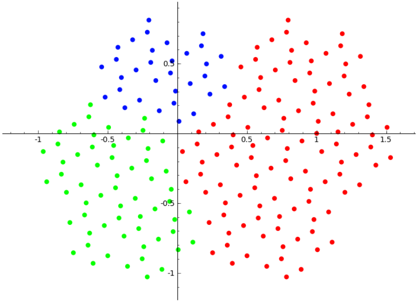

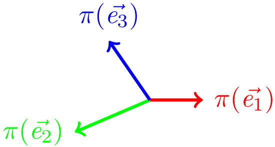

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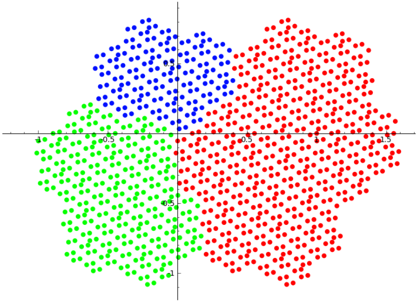

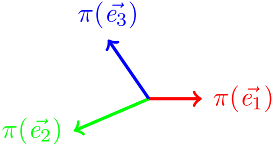

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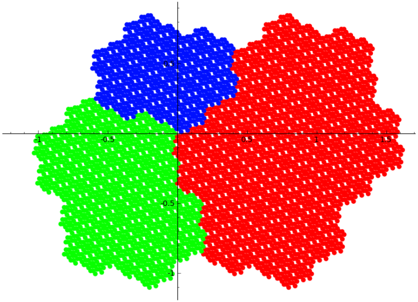

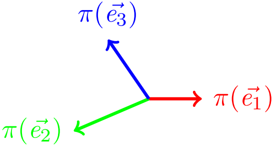

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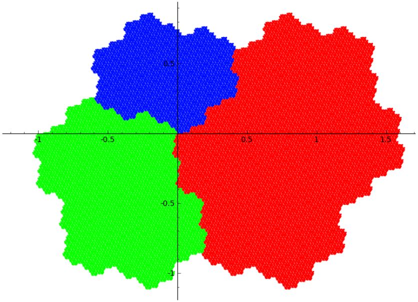
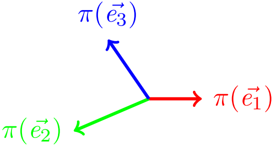




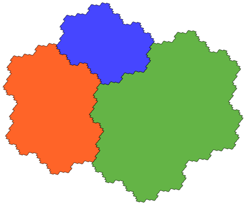
Pisot numbers, codings and fractals

$$X^3 = X^2 + X + 1$$

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

Theorem [Rauzy'82] The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the translation T_β on the two-dimensional torus $\mathbb{T}^2$

$$T_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$



Beyond the Pisot conjecture

Classical exponentially convergent multidimensional continued fraction algorithms generate faithful symbolic codings for translations on the torus.

Take your favourite algorithm A.

Theorem [B.-Steiner-Thuswaldner, Pytheas Fogg-Noûs]
For almost every $\boldsymbol{\alpha} \in [0,1]^d$, the translation $T_{\boldsymbol{\alpha}} : \mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\alpha}$ on the torus $\mathbb{T}^d$ admits a symbolic model: the S-adic system provided by the multidimensional continued fraction algorithm A applied to $\boldsymbol{\alpha}$ is isomorphic in measure to $T_{\boldsymbol{\alpha}}$. Moreover, factors have bounded discrepancy.

And now?

The discrepancy is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_{a} \sup_{n\in\mathbb{N}} \left|\mathrm{Card}\{k, 0 \le k \le n-1, u_k = a\} - n\alpha_a\right|$$

One has

$$\sup_{\boldsymbol{\alpha}} \inf_{u} \Delta_{\boldsymbol{\alpha}}(u) = 1 - \frac{1}{2d-2}$$

Now, given $\boldsymbol{\alpha}$, what about

$$\inf_{u} \Delta_{\boldsymbol{\alpha}}(u)?$$